MAT 22B - Lecture Notes

2 September 2015

Systems of ODEs

This is some really cool stuff. Systems of ODEs are a natural and powerful way to model how multiple different interrelated quantities change through time. As a first example, we can think about tanks with solutions flowing in and out of them.

Say you have two tanks that contain a solution of salt. Each tank has a capacity of 100 liters. A solution containing 3 mol/L of salt is being poured into tank 1 at a rate of 5 L/min, and a solution containing 2 mol/L of salt is being poured into tank 2 at a rate of 4 L/min. Moreover, the tanks are mixing with each other; the solution in tank 1 flows into tank 2 at a rate of 10 L/min, and the solution in tank 2 flows into tank 1 at a rate of 10 L/min. To keep the total amount of fluid constant, solution drains from tank 1 at a rate of 5 L/min and from tank 2 at a rate of 4 L/min.

We can describe this system's behavior through time by a *pair* of functions, $Q_1(t)$ and $Q_2(t)$, representing the amount of salt in tank 1 and tank 2, respectively, at time t. From the description, we can write down an ODE for Q_1 and Q_2. First,

$$\frac{dQ_1}{dt} = \left(3\frac{\text{mol}}{\text{L}}\right)\left(5\frac{\text{L}}{\text{min}}\right) + \left(10\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_2}{100}\frac{\text{mol}}{\text{L}}\right) - \left(15\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_1}{100}\frac{\text{mol}}{\text{L}}\right)$$

the terms above represent the solution being poured in, the solution coming from tank 2, and the drainage, respectively. Likewise, for tank 2,

$$\frac{dQ_2}{dt} = \left(2\frac{\text{mol}}{\text{L}}\right)\left(4\frac{\text{L}}{\text{min}}\right) + \left(10\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_1}{100}\frac{\text{mol}}{\text{L}}\right) - \left(14\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_2}{100}\frac{\text{mol}}{\text{L}}\right)$$

dropping the units for brevity, we have

$$\begin{aligned} Q_1' &= 15 + \frac{1}{10}Q_2 - \frac{15}{100}Q_1 \\ Q_2' &= 8 + \frac{1}{10}Q_1 - \frac{14}{100}Q_2 \end{aligned}$$

We can also write this as a matrix equation:

$$\frac{d}{dt}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} -\frac{15}{100} & \frac{1}{10} \\ \frac{1}{10} & \frac{-14}{100} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 15 \\ 8 \end{pmatrix}$$

another way to simplify notation is to simply write

$$\vec{Q}(t) = \begin{pmatrix} Q_1(t) \\ Q_2(t) \end{pmatrix}$$

and then our system reads

$$\vec{Q}' = A\vec{Q} + \vec{b}$$

where $A = \begin{pmatrix} -\frac{15}{100} & \frac{1}{10} \\ \frac{1}{10} & \frac{-14}{100} \end{pmatrix}$ and $\vec{b} = \begin{pmatrix} 15 \\ 8 \end{pmatrix}$. So really, this system of two differential equations obeyed by two real-valued functions (Q_1 and Q_2) can be thought of as a single differential equation obeyed by a single vector-valued function $\vec{Q}$. This consideration does not depend on the specifics of this problem, but rather will be the way we think about all systems of ODEs.

Every ODE is secretly first order

Another way that systems of ODEs can arise is as a way to represent a second order ODE. For example, consider the second order ODE

$$x'' + 2x' - 3x = 0$$

If we define a new variable $y(t) = x'(t)$, then we have

$$y'(t) = x''(t) = 3x(t) - 2x'(t) = 3x - 2y$$

So the original second order ODE is equivalent to the system

$$\begin{aligned} x'(t) &= y(t) \\ y'(t) &= 3x(t) - 2y(t) \end{aligned}$$

or, in matrix form

$$\frac{d}{dt}\begin{pmatrix} x \\ y \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ 3 & -2 \end{pmatrix}\begin{pmatrix} x \\ y \end{pmatrix}$$

To clarify what I mean about "equivalent": I mean that if $x(t)$ is a solution to $x''+2x'-3x = 0$, then the vector-valued function $\vec{x}(t) := \begin{pmatrix} x(t) \\ y(t) \end{pmatrix} = \begin{pmatrix} x(t) \\ x'(t) \end{pmatrix}$ satisfies the vector ODE above, and vice versa. All we've done is *rewritten* the equation. This is on par with $ax^2+bx+c = 0$ vs. $x = \frac{1}{2a}\left(-b \pm \sqrt{b^2 - 4ac}\right)$.

A key thing to notice aboout the system is that now it is a *first order* equation, since it only contains a first derivative. This makes the system easier to visualize, and also possible to solve numerically using, for instance, Euler's method (the tangent line method). Another feature of the system that we'll refer to is its "dimension". If you write the system as a list of equations, the dimension is the number of equations. If you write it as a single equation for a vector-valued function, the dimension of the system is the number of components in those vector values.

This is another feature that is quite general - any ODE of *any order* can be rewritten as a first order system of ODEs. For example, a third order equation

$$x''' - x'' - x' - x = 0$$

can be written as a first order system by defining $x_1 = x$, $x_2 = x_1'$ and $x_3 = x_2'$. Then $x_3' = x_2'' = x''' = x + x' + x'' = x_1 + x_2 + x_3$. In summary,

$$\begin{aligned} x_1' &= x_2 \\ x_2' &= x_3 \\ x_3' &= x_1 + x_2 + x_3 \end{aligned}$$

Or, if we let $\vec{x} = (x_1, x_2, x_3)^\top$, then

$$\vec{x}' = \begin{pmatrix} 0 & 1 & 0 \\ 0 & 0 & 1 \\ 1 & 1 & 1 \end{pmatrix} \vec{x}$$

You can convince yourself that an n^{th} order equation (for a scalar-valued function $y(t)$) can be transformed in exactly this way into a system of n first order equations - or equivalently, as a first order ODE for a function which takes values in $\mathbb{R}^n$. This fact helps elucidate the principle that to specify a specific solution to an n^{th} order ODE, we need n initial conditions $(y(0), y'(0), \ldots, y^{(n-1)}(0))$. When we view the n^{th} order equation as a first order equation for a vector-valued function, like

$$\vec{x}'(t) = \vec{F}(\vec{x}, t)$$

then we should need to specify only one initial condition, $\vec{x}(0)$. Of course, this initial condition should be a vector with n components, and so it is given by n scalar values, which we would have needed had we kept the problem as a single n^{th} order equation.

Example

Let's see what this looks like a bit more concretely. We'll convert the equation for an undamped mass on a spring into a first order system. The ODE we started with is

$$my'' + ky = 0$$

Now I'll define two functions: $y_1(t) = y(t)$, and $y_2(t) = y'(t)$. Then y_2 satisfies the ODE

$$y_2' = y_1'' = -\frac{k}{m} y = -\frac{k}{m} y_1$$

So the vector-valued function

$$\vec{y}(t) = \begin{pmatrix} y_1(t) \\ y_2(t) \end{pmatrix}$$

satisfies the equation

$$\vec{y}'(t) = \begin{pmatrix} 0 & 1 \\ -\frac{k}{m} & 0 \end{pmatrix} \vec{y}$$

Now, we know the solutions from previous work. For example, $y(t) = \sin\left(\sqrt{k/m}t\right)$ is a solution. Turning this into the vector $\vec{y}(t)$ as defined above we get

$$\vec{y}(t) = \begin{pmatrix} \sin\left(\sqrt{k/m}t\right) \\ \sqrt{k/m}\cos\left(\sqrt{k/m}t\right) \end{pmatrix}$$

and an easy calculation shows that this vector-valued function satisfies the ODE above.

Linear vs. Nonlinear Systems

Just as in the scalar-valued case (what we've been doing so far), we can classify systems of ODEs as linear or nonlinear, homogeneous or nonhomogeneous, and so on. To simplify notation, we'll restrict our attention to first order systems. Note that because of the above discussion, this does not lose us any generality - any ODE you can think of can be written as a first order system, if you include enough dependent variables.

Linear

A first-order system of ODE is called *linear* if it is of the form

$$\frac{d\vec{y}}{dt} + \mathbf{A}(t)\vec{y} = \vec{g}(t)$$

where $\mathbf{A}(t)$ is an $n \times n$ matrix (which may depend on time), where n is the number of components of $\vec{y}$, and $\vec{g}(t)$ is another n-dimensional vector which may depend on time. If $\mathbf{A}(t)$ is actually independent of t, we say the equation has *constant coefficients*. If the right-hand side $\vec{g}(t)$ is equal to $\vec{0}$ for all t, then we say that the equation is *homogeneous*. If we write out the above equation in components, we have

$$\begin{aligned} y_1' + a_{11}(t)y_1(t) + a_{12}(t)y_2(t) + \cdots + a_{1n}y_n(t) &= g_1(t) \\ y_2' + a_{21}(t)y_1(t) + a_{22}(t)y_2(t) + \cdots + a_{2n}y_n(t) &= g_2(t) \\ \vdots \quad &\vdots \quad \vdots \\ y_n' + a_{n1}(t)y_1(t) + a_{n2}(t)y_2(t) + \cdots + a_{nn}y_n(t) &= g_n(t) \end{aligned}$$

I hope you understand why I prefer the matrix notation.

As was the case for scalar-valued ODEs, a homogeneous system with constant coefficients is a very nice starting point (i.e. there are many closed-form solutions to be had). As a warm up, consider the system

$$\frac{d\vec{y}}{dt} - \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} \vec{y} = 0$$

In components, this reads

$$\begin{aligned} y_1' - y_1 &= 0 \\ y_2' + y_2 &= 0 \end{aligned}$$

We can solve each of those separately in a snap; $y_1 = c_1 e^{-t}$ and $y_2 = c_2 e^t$, so the general solution is

$$\vec{y}(t) = \begin{pmatrix} c_1 e^t \\ c_2 e^{-t} \end{pmatrix}$$

What made this so easy is that the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ is *diagonal*, so that y_1 only "talks to" y_1 and y_2 only "talks to" y_2. A thing to notice about diagonal matrices is that their entries are their eigenvalues, and the standard basis vectors are their eigenvectors. Moreover, it was these diagonal entries that determined the exponents in our solution!

Let's look at a more general picture. Say I have a system of ODE which is

$$\frac{d\vec{y}}{dt} = \mathbf{A}y$$

this is linear, homogeneous, and with constant coefficients. Now suppose $\vec{y}_0$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. That is, $\mathbf{A}\vec{y}_0 = \lambda\vec{y}_0$. Now, consider the vector-valued function

$$\vec{y}(t) = \vec{y}_0 e^{\lambda t}$$

If you were to visualize this vector-valued function, it would simply move along the ray spanned by the vector $\vec{y}_0$ in a way determined by λ. Let's see if this is a solution to our ODE. First compute its derivative:

$$\frac{d\vec{y}}{dt} = \frac{d}{dt}\left(\vec{y}_0 e^{\lambda t}\right) = \lambda\vec{y}_0 e^{\lambda t}$$

and now the right-hand side

$$\mathbf{A}\vec{y}(t) = \mathbf{A}\vec{y}_0 e^{\lambda t} = \lambda\vec{y}_0 e^{\lambda t}$$

OMG! They're the same! What a gosh darned coincidence.

This works for any eigenvector/eigenvalue pair of the matrix $\mathbf{A}$. So, if $(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n)$ are eigenvectors of $\mathbf{A}$ with eigenvalues $(\lambda_1, \lambda_2, \ldots, \lambda_n)$, then each function $\vec{y}_k e^{\lambda_k t}$ is a solution. By linearity and homegeneity of the ODE, then, we have that for any $c_1, c_2, \ldots, c_n$, the function

$$c_1\vec{y}_1 e^{\lambda_1 t} + \cdots + c_n\vec{y}_n e^{\lambda_n t}$$

is also a solution. Notice that if some of the eigenvalues are repeated, we've "overcounted" how many degrees of freedom there are in the above expression. Likewise, if some of the eigenvalues are complex, we have to convert the complex exponentials back into real functions, which will give us sines and cosines.

This should all sound very reminiscent of the procedure we went through in solving the characteristic equation, because really, this is what we were doing. Indeed, for the equation

$$ay'' + by' + cy = 0$$

we can convert to a 2D system by setting $y_1 = y$ and $y_2 = y'$, so that $y_2' = y'' = -{}^{b}\!/\!{}_{a}y' - {}^{c}\!/\!{}_{a}y$. So,

$$\begin{aligned} y_1' &= y_2 \\ y_2' &= -\frac{c}{a}y_1 - \frac{b}{a}y_2 \end{aligned}$$

In matrix form, we have

$$\frac{d\vec{y}}{dt} = \begin{pmatrix} 0 & 1 \\ -c/a & -b/a \end{pmatrix} \vec{y}$$

The eigenvalues of the above matrix are the solutions λ to the equation $\det(\mathbf{A} - \lambda\mathbf{I}) = 0$, which reads

$$-\lambda\left(-b/a - \lambda\right) + \frac{c}{a} = 0$$

or

$$\lambda^2 + \frac{b}{a}\lambda + \frac{c}{a} = 0 \iff a\lambda^2 + b\lambda + c = 0$$

Wouldja lookit that.

Nonlinear Systems

This is where things get REALLY interesting. We'll get here next time and talk about how every ODE is secretly just a vector field on some phase space. This is where studying differential equations becomes much less about formulae and much more about drawing good pictures.