MAT 22B - Lecture Notes

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Solving IVP's using the Laplace Transform

As we discussed last time, the Laplace transform $\mathcal{L}$ is an operator that eats functions and spits out other functions, according to the formula

$$\mathcal{L}\{y\} = \int_0^\infty e^{-st} y(t) dt := Y(s)$$

It is useful in solving differential equations because of the identity

$$\mathcal{L}\{y'\} = s\mathcal{L}\{y\} - y(0)$$

which can be derived using integration by parts. Sometimes the way we talk about this property is to say that the Laplace transform "turns differentiation into multiplication by s". If you want to know how fancy this can get, come talk to me. Using the above formula, we can get similar formulas for a derivative of any order. In particular, we get

$$\mathcal{L}\{y''\} = s^2\mathcal{L}\{y\} - sy(0) - y'(0)$$

and in general,

$$\mathcal{L}\{y^{(n)}\} = s^n\mathcal{L}\{y\} - s^{n-1}y(0) - \cdots - sy^{(n-2)}(0) - y^{(n-1)}(0)$$

For us, the general picture is:

$$y(t) \text{ satisfies a differential equation} \iff \mathcal{L}\{y\} = Y(s) \text{ satisfies an algebraic equation}$$

So, given a differential equation for y, we can use the Laplace transform to find out an algebraic equation that its transform $Y(s)$ satisfies. Notice that this step depends on the initial conditions $y(0)$ and $y'(0)$, because of the formulas for $\mathcal{L}\{y'\}$ and $\mathcal{L}\{y''\}$. Then we can solve this equation using the algebra you learned in high school to get $Y(s)$, and finally "invert the transform" - that is, find $y(t)$ so that $\mathcal{L}\{y\} = Y$. This $y(t)$ will be the solution to the DE we started with.

Inverting the Transform

This is the trickiest step in using the LT to solve IVP's. While applying the transform can be done using the tidy formula we saw at the beginning, there's no such tidy formula for undoing the transform. As such, our methods will seem rather clunky. The basic idea is that once you have $Y(s)$, you should manipulate it until it's broken up into pieces that you can recognize as the Laplace transforms of certain functions. For instance, I know that

$$\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$$

So if I transform an IVP and find out that, for instance, $Y(s) = \frac{1}{s-1}$, I know that $y(t) = e^t$. This same logic applies for all different sorts of functions - a good table can be found on page 321 in the text, or in any number of places online.

Now let's get our hands dirty.

Transforms of some basic functions

Exponentials $y = e^{at}$

$$\begin{aligned}
\mathcal{L}\{e^{at}\} &= \int_0^\infty e^{-st}e^{at}dt \\
&= \int_0^\infty e^{(a-s)t}dt \\
&= \frac{1}{a-s}e^{(a-s)t}\Big|_0^\infty \\
&= \frac{1}{s-a}
\end{aligned}$$

Sines $y = \sin(at)$

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)dt$$

integrating by parts with $u = \sin(at)$ and $dv = e^{-st}$, we get

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \frac{-1}{s}e^{-st}\sin(at) - \int_0^\infty \frac{-a}{s}e^{-st}\cos(at)dt \\
&= \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt
\end{aligned}$$

integrating by parts again wth $u = \cos(at)$ and $dv = e^{-st}$, we get

$$\begin{aligned}
\mathcal{L}\{\sin(at)\} &= \frac{a}{s}\left[\frac{-1}{s}e^{-st}\cos(at) - \int_0^\infty \frac{a}{s}e^{-st}\sin(at)dt\right] \\
&= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\mathcal{L}\{\sin(at)\}\right]
\end{aligned}$$

solving for $\mathcal{L}\{\sin(at)\}$, we get

$$\left(1 + \left(\frac{a}{s}\right)^2\right)\mathcal{L}\{\sin(at)\} = \frac{a}{s^2} \implies \mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

And I'll leave the cosine calculation to you.

Products of exponentials with trig functions, $y = e^{at}\sin(bt)$

$$\begin{aligned}
\mathcal{L}\{e^{at}\sin(bt)\} &= \int_0^\infty e^{-st}e^{at}\sin(bt)dt \\
&= \int_0^\infty e^{(a-s)t}\sin(bt)dt
\end{aligned}$$

at this point we can pause and notice a relationship to $\mathcal{L}\{\sin(bt)\}$. If we let $Y(s) = \mathcal{L}\{\sin(bt)\} = \frac{b}{s^2+b^2}$, then we have

$$\begin{aligned}
Y(s-a) &= \int_0^\infty e^{-(s-a)t}\sin(bt)dt \\
&= \int_0^\infty e^{(a-s)t}\sin(bt)dt = \mathcal{L}\{e^{at}\sin(bt)\}(s)
\end{aligned}$$

So we have

$$\mathcal{L}\{e^{at}\sin(bt)\} = Y(s-a) = \frac{b}{(s-a)^2 + b^2}$$

A similar trick will get you from $\mathcal{L}\{\cos(bt)\}$ to $\mathcal{L}\{e^{at}\cos(bt)\}$.

Polynomials: $y = a_n t^n + \cdots + a_1 t + a_0$. Because $\mathcal{L}$ is linear, we just need to figure out $\mathcal{L}\{t^n\}$ for any n. Let's start with $n = 0$.

$$\begin{aligned}
\mathcal{L}\{t^0\} &= \mathcal{L}\{1\} \\
&= \int_0^\infty e^{-st}dt \\
&= \frac{-1}{s}e^{-st}|_0^\infty = \frac{1}{s}
\end{aligned}$$

Easy enough. Then $n = 1$ requires integration by parts (we'll take $u = t$ and $dv = e^{-st}dt$:

$$\begin{aligned}
\mathcal{L}\{t^1\} &= \int_0^\infty te^{-st}dt \\
&= \frac{-1}{s}te^{-st}|_0^\infty - \int_0^\infty \frac{-1}{s}e^{-st}dt \\
&= \frac{1}{s}\int_0^\infty e^{-st}dt = \frac{1}{s^2}
\end{aligned}$$

Notice that integration by parts was able to reduce the power of t by one, which let us use the fact that we knew the transform of t^0. This is a clue that we should be able to get a formula for $\mathcal{L}\{t^n\}$ in terms of $\mathcal{L}\{t^{n-1}\}$. Let's see what I mean:

$$\mathcal{L}\{t^n\} = \int_0^\infty t^n e^{-st}dt$$

integrating by parts with $u = t^n$ and $dv = e^{-st}dt$, we get

$$\begin{aligned}
\mathcal{L}\{t^n\} &= \frac{-1}{s}t^n e^{-st}|_0^\infty - \int_0^\infty \frac{-1}{s}nt^{n-1}e^{-st}dt \\
&= \frac{n}{s}\int_0^\infty t^{n-1}e^{-st}dt \\
&= \frac{n}{s}\mathcal{L}\{t^{n-1}\}
\end{aligned}$$

Nice! So, for instance, $\mathcal{L}\{t^2\} = \frac{2}{s}\mathcal{L}\{t\} = \frac{2}{s^3}$, $\mathcal{L}\{t^3\} = \frac{3}{s}\mathcal{L}\{t^2\} = \frac{6}{s^4}$, and so on. The general formula is

$$\mathcal{L}\{t^n\} = \frac{n!}{s^{n+1}}$$

Examples

At this point we have enough firepower to start solving some interesting problems.

Let's start slow. Consider the first-order linear IVP $y' + 2y = 0$, $y(0) = 1$. Then if $Y(s)$ is the Laplace transform of y, we have

$$\begin{aligned}
\mathcal{L}\{y' + 2y\} &= \mathcal{L}\{0\} \\
\mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \int_0^\infty e^{-st} \cdot 0dt \\
sY(s) - y(0) + 2Y(s) &= 0
\end{aligned}$$

which is now an *algebraic* equation for Y! Notice also that we *need* the initial condition $y(0)$ in the above equation. Since we do have $y(0)=1$, we can solve for Y and get

$$Y(s) = \frac{1}{s+2}$$

So, comparing to the formulas we got above, we find $y(t)=e^{-2t}$. Notice that there are no free parameters in this answer - $y(t)=e^{-2t}$ on the nose. This is because the determination of $Y(s)$ incorporated the initial condition already.

We can see what happens if we impose a different initial condition, say $y(0)=y_0$. Then the algebraic equation satisfied by $Y(s)$ is

$$sY(s) - y_0 + 2Y(s) = 0$$

which gives

$$Y(s) = \frac{y_0}{s+2}$$

Now, by linearity of the Laplace transform, we have that

$$\mathcal{L}\{y_0 e^{at}\} = y_0 \mathcal{L}\{e^{at}\} = \frac{y_0}{s-a}$$

so in our case, $y(t) = y_0 e^{-2t}$, as we knew it had to be.

Let's check out a second order ODE. This is problem 12 in section 6.2.

$$y'' + 3y' + 2y = 0, \quad y(0) = 1,\ y'(0) = 0$$

Taking the LT of both sides we get

$$\begin{aligned}
\mathcal{L}\{y''\} + 3L\{y'\} + 2\mathcal{L}\{y\} &= 0 \\
\left(s^2 Y - sy(0) - y'(0)\right) + 3\left(sY - y(0)\right) + 2\left(Y\right) &= 0 \\
\left(s^2 + 3s + 2\right) Y - s - 3 &= 0
\end{aligned}$$

where we have used $y(0)=1$ and $y'(0)=0$. Solving for $Y(s)$ gives

$$Y(s) = \frac{s+3}{s^2+3s+2} = \frac{s+3}{(s+2)(s+1)}$$

Now to get Y into the form of some things that we recognize, we can break up the above expression using partial fractions. We have

$$Y = \frac{A}{s+2} + \frac{B}{s+1}$$

Clearing the denominators gives

$$s + 3 = (s+1)A + (s+2)B$$

which gives $A=-1$ and $B=2$, so

$$Y(s) = \frac{-1}{s+2} + \frac{2}{s+1}$$

comparing to our existing knowledge and using linearity of $\mathcal{L}$, we find then that $y(t) = -e^{-2t} + 2e^{-t}$

We can even use the LT to solve higher order linear equations with constant coefficients! This example is problem 18 in section 6.2

$$y^{(4)} - y = 0; \quad y(0) = 1,\ y'(0) = 0,\ y''(0) = 1,\ y'''(0) = 0$$

Taking $\mathcal{L}$ of both sides gives

$$\begin{aligned} \mathcal{L}\{y^{(4)}\} - \mathcal{L}\{y\} &= 0 \\ s^4Y - s^3y(0) - s^2y'(0) - sy''(0) - y'''(0) - Y &= 0 \\ (s^4 - 1)Y - s^3 - s &= 0 \end{aligned}$$

so

$$Y(s) = \frac{s(s^2+1)}{s^4-1} = \frac{s(s^2+1)}{(s^2-1)(s^2+1)} = \frac{s}{s^2-1} = \frac{s}{(s+1)(s-1)}$$

Now using partial fractions

$$Y(s) = \frac{A}{s+1} + \frac{B}{s-1} \iff s = A(s-1) + B(s+1)$$

which gives us $A = \frac{1}{2}$ and $B = \frac{1}{2}$, so

$$Y = \frac{1/2}{s-1} + \frac{1/2}{s+1}$$

and finally,

$$y(t) = \frac{1}{2}\left(e^t + e^{-t}\right) = \cosh(t)$$

This is the *hyperbolic cosine* function. It's a cool function!

Next, let's consider the general case of a second-order, constant coefficient, linear ODE. To start, we'll consider the homogeneous case. That is, we're considering ODEs of the form

$$ay'' + by' + cy = 0$$

Applying the Laplace transform gives

$$\begin{aligned} \mathcal{L}\{ay'' + by' + cy\} &= \mathcal{L}\{0\} \\ a\mathcal{L}\{y''\} + b\mathcal{L}\{y'\} + c\mathcal{L}\{y] &= 0 \\ a\left(s^2Y - sy(0) - y'(0)\right) + b\left(sY - y(0)\right) + cY &= 0 \\ \left(as^2 + bs + c\right)Y - (as+b)y(0) - ay'(0) &= 0 \end{aligned}$$

So we get

$$Y(s) = \frac{(as+b)\,y(0) + ay'(0)}{as^2 + bs + c}$$

Now to find $y(t)$ we should just match the function above to the form of something we recognize.

OK, hold the phone. That function looks awful. But we know that solutions to this equation should be things like e^{rt}, $e^{\lambda t}\sin(\mu t)$, or maybe te^{rt}. Those things have much tidier-looking Laplace transforms than that. What gives?

Remember partial fractions? Me neither.

The key step here in simplifying our expression for $Y(s)$ is, as you could have guessed, factoring the polynomial $as^2 + bs + c$. Again, we encounter the characteristic polynomial. If the roots are real and distinct, say s_1, s_2, we have

$$Y = \frac{A}{s - s_1} + \frac{B}{s - s_2}$$

for some constants A and B to be solved for, and then $y(t) = Ae^{s_1 t} + Be^{s_2 t}$

If the roots are complex, however, then $as^2 + bs + c$ is what we call an *irreducible quadratic.* In this case, the best we can do is complete the square. For simplicity, take $a = 1$. Then

$$\begin{aligned} as^2 + bs + c &= s^2 + bs + c \\ &= \left(s + \frac{b}{2}\right)^2 + c - \frac{b^2}{4} \end{aligned}$$

Since the quadratic is irreducible, we know that $b^2 - 4c < 0$, or $b^2/4 < c$, so that $c - b^2/4 > 0$, and so we can write

$$s^2 + bs + c = \left(s + \frac{b}{2}\right)^2 + \mu^2$$

where $\mu = \sqrt{c - b^2/4}$, and then our expression for $Y(s)$ is

$$Y(s) = \frac{As + B}{\left(s + \frac{b}{2}\right)^2 + \mu^2}$$

for some A and B. Then by the LT tables, we get

$$y(t) = Ae^{\frac{-b}{2}t}\cos(\mu t) + Be^{\frac{-b}{2}t}\sin(\mu t)$$

Compare this to the answer you'd get from finding the roots of the characteristic equation.

Systems of ODEs

This is some really cool stuff. Systems of ODEs are a natural and powerful way to model how multiple different interrelated quantities change through time. As a first example, we can think about tanks with solutions flowing in and out of them.

Say you have two tanks that contain a solution of salt. Each tank has a capacity of 100 liters. A solution containing 3 mol/L of salt is being poured into tank 1 at a rate of 5 L/min, and a solution containing 2 mol/L of salt is being poured into tank 2 at a rate of 4 L/min. Moreover, the tanks are mixing with each other; the solution in tank 1 flows into tank 2 at a rate of 10 L/min, and the solution in tank 2 flows into tank 1 at a rate of 10 L/min. To keep the total amount of fluid constant, solution drains from tank 1 at a rate of 5 L/min and from tank 2 at a rate of 4 L/min.

We can describe this system's behavior through time by a *pair* of functions, $Q_1(t)$ and $Q_2(t)$, representing the amount of salt in tank 1 and tank 2, respectively, at time t. From the description, we can write down an ODE for Q_1 and Q_2. First,

$$\frac{dQ_1}{dt} = \left(3\frac{\text{mol}}{\text{L}}\right)\left(5\frac{\text{L}}{\text{min}}\right) + \left(10\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_2}{100}\frac{\text{mol}}{\text{L}}\right) - \left(15\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_1}{100}\frac{\text{mol}}{\text{L}}\right)$$

the terms above represent the solution being poured in, the solution coming from tank 2, and the drainage, respectively. Likewise, for tank 2,

$$\frac{dQ_2}{dt} = \left(2\frac{\text{mol}}{\text{L}}\right)\left(4\frac{\text{L}}{\text{min}}\right) + \left(10\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_1}{100}\frac{\text{mol}}{\text{L}}\right) - \left(14\frac{\text{L}}{\text{min}}\right)\left(\frac{Q_2}{100}\frac{\text{mol}}{\text{L}}\right)$$

dropping the units for brevity, we have

$$\begin{aligned} Q_1' &= 15 + \frac{1}{10}Q_2 - \frac{15}{100}Q_1 \\ Q_2' &= 8 + \frac{1}{10}Q_1 - \frac{14}{100}Q_2 \end{aligned}$$

We can also write this as a matrix equation:

$$\frac{d}{dt}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} = \begin{pmatrix} -\frac{15}{100} & \frac{1}{10} \\ \frac{1}{10} & \frac{-14}{100} \end{pmatrix}\begin{pmatrix} Q_1 \\ Q_2 \end{pmatrix} + \begin{pmatrix} 15 \\ 8 \end{pmatrix}$$