MAT 22B - Lecture Notes

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The Laplace Transform

The Laplace transform is a really neat mathematical tool that can be used to solve initial value problems, among other things. One of its important advantages is that it can deal with forcing functions (i.e. nonhomogeneous terms) that are discontinuous, or even "spikey" like the delta function (a function which is nonzero at only a single point but has a nonzero integral).

Besides that, the Laplace transform is a good thing to see because it illustrates the notion of an *integral transform*, and how they can be incredibly useful in simplifying hard problems.

The Laplace transform is a transform. That is, it's a machine that takes input and gives output. The input it takes is a function $f(t)$ (with certain nice properties to be specified soon). The output is also a function, but of a different variable. Typically the way we write this is

$$\mathcal{L}\{f(t)\} = F(s)$$

I find that sometimes including the (t) and (s) are confusing, so I'll write

$$\mathcal{L}\{f\} = F$$

where f is a function of t and F is a function of s. My point in writing it this way is that you give $\mathcal{L}$ the *whole* function f, and it spits you out the whole function F. We can, of course, evaluate either f or F at some point of our choosing, but that's secondary.

The big picture of how the Laplace transform is used is that if you are looking for a function f that satisfies a certain ODE, you can determine a *different* equation that its Laplace transform F satisfies. In many cases, this latter equation will contain no derivatives at all! That means it's super simple to solve for F, and then we recover f by *inverting* the transform (this is typically the hard part).

The actual formula relating a function f to its Laplace transform F is quite simple:

$$F(s) = \int_0^\infty e^{-st} f(t)dt$$

For each different value of s, the value of F is given by the integral of a different function. Maybe that last sentence was obvious, or maybe it was still gibberish - either way, it needs to be said.

At this point we can see the conditions that f needs to meet - the integral defining F is an improper integral, so f should be nice enough that the integral converges for at least some values of s. That's really it, and the book gives some concrete criteria to check, though it's easy to get lost in the mess of "there exists" and "for all". Suffice it to say that f is piecewise continuous and is eventually bounded in size by e^{at} for some a (this last property is called being "of exponential order"), and in this case the Laplace transform $F(s)$ is defined for all $s > a$. A great many functions satisfy these properties, and you might be hard-pressed to find one that doesn't - an example of a function not of exponential order is $y = e^{t^2}$.

So, we've got this gizmo that turns functions into other functions. What good is it? Great question. For one, it's a *linear operator*. Just look - if f_1 and f_2 are two functions and c_1 and c_2 are constants, then

$$\begin{aligned}
\mathcal{L}\{c_1 f_1 + c_2 f_2\}(s) &= \int_0^\infty e^{-st}\left(c_1 f_1(t) + c_2 f_2(t)\right) dt \\
&= c_1 \int_0^\infty e^{-st} f_1(t)dt + c_2 \int_0^\infty e^{-st} f_2(t)dt \\
&= c_1 \mathcal{L}\{f_1\}(s) + c_2 \mathcal{L}\{f_2\}(s)
\end{aligned}$$

This property means that it will "play nicely" with our linear differential equations.

The next most important property is the way it deals with derivatives. Say f is a differentiable function and f' is its derivative. It turns out we can express the Laplace transform of f' in terms of the Laplace transform of f! The key is integration by parts:

$$\mathcal{L}\{f'\} = \int_0^\infty e^{-st} f'(t)dt$$

taking $u = e^{-st}$ and $dv = f'(t)dt$, we get $v = f(t)$ and $du = -se^{-st}dt$, so

$$\begin{aligned}
\mathcal{L}\{f'\} &= e^{-st} f(t)|_0^\infty - \int_0^\infty (-se^{-st}) f(t)dt \\
&= e^{-st} f(t)|_0^\infty + s\int_0^\infty e^{-st} f(t)dt \\
&= -f(0) + s\mathcal{L}\{f\}
\end{aligned}$$

Let me step more clearly through the equality $e^{-st}f(t)|_0^\infty = -f(0)$. Formally, we can't just "plug in $t = \infty$", so we should take a limit. Then

$$e^{-st} f(t)|_0^\infty = \left(\lim_{t\to\infty} e^{-st} f(t)\right) - f(0)$$

but we've assumed f has a Laplace transform. So, it is eventually bounded by e^{at} for some a, and that s should be greater than this a. It follows then that e^{-st} decays to zero faster than $f(t)$ might go off to infinity, so $\left(\lim_{t\to\infty} e^{-st}f(t)\right) = 0$.

So what we've found is

$$\mathcal{L}\{f'\} = s\mathcal{L}\{f\} - f(0)$$

Immediately we can see that this fact can help up solve IVP's. Consider the first-order linear IVP $y' + 2y = 0$, $y(0) = 1$. Then if $Y(s)$ is the Laplace transform of y, we have

$$\begin{aligned} \mathcal{L}\{y' + 2y\} &= \mathcal{L}\{0\} \\ \mathcal{L}\{y'\} + 2\mathcal{L}\{y\} &= \int_0^\infty e^{-st} \cdot 0 dt \\ sY(s) - y(0) + 2Y(s) &= 0 \end{aligned}$$

which is now an *algebraic* equation for Y! Notice also that we *need* the initial condition $y(0)$ in the above equation. Since we do have $y(0) = 1$, we can solve for Y and get

$$Y(s) = \frac{1}{s+2}$$

and now "all we have to do" is invert the transform to get $y(t)$.

Inverting the Transform

This is the tough part. While there's a nice tidy formula for $\mathcal{L}\{f\}$ for whatever f your heart desires, there is no such tidy formula for $\mathcal{L}^{-1}\{F\}$ (at least not within the scope of this class or my knowledge). The way we find inverses in practice is to use a table. You solve for $Y(s)$, and then mush it around and rearrange it until it looks like something you recognize. For example, if $f(t) = e^{-2t}$, I can compute its Laplace transform:

$$\begin{aligned} \mathcal{L}\{e^{-2t}\} &= \int_0^\infty e^{-st}e^{-2t} dt \\ &= \int_0^\infty e^{(-2-s)t} dt \\ &= \frac{1}{-2-s} e^{(-2-s)t}\Big|_0^\infty \\ &= \frac{1}{s+2} \end{aligned}$$

provided that $s > -2$ so that the integral converges. Now I know that whenever I see a function whose Laplace transform is $\frac{1}{s+2}$, the original function was e^{-2t}. This case happened to be the answer to the IVP we just saw, because I made it that way. Other cases are dealt with similarly.

The simplest cases are $f(t) = e^{rt}$. By the same sort of calculation, $\mathcal{L}\{e^{rt}\} = \frac{1}{s-r}$, but only for $s > r$, so that the integral defining the transform converges.

We can also consider higher derivatives, using the same rule we found out for the first derivative. We have

$$\begin{aligned}\mathcal{L}\{f''\} &= s\mathcal{L}\{f'\} - f'(0) \\ &= s\left(s\mathcal{L}\{f\} - f(0)\right) - f'(0) \\ &= s^2\mathcal{L}\{f\} - sf(0) - f'(0)\end{aligned}$$

You can convince yourself that for the n-th derivative, you get

$$\mathcal{L}\{f^{(n)}\} = s^n\mathcal{L}\{f\} - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

This is a good place to practice proof by induction if you're a mathematician or computer scientist.

Another useful class of functions whose Laplace transforms we'd like to know is trig functions. For example,

$$\mathcal{L}\{\sin(at)\} = \int_0^\infty e^{-st}\sin(at)dt$$

integrating by parts with $u = \sin(at)$ and $dv = e^{-st}$, we get

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= \frac{-1}{s}e^{-st}\sin(at) - \int_0^\infty \frac{-a}{s}e^{-st}\cos(at)dt \\ &= \frac{a}{s}\int_0^\infty e^{-st}\cos(at)dt\end{aligned}$$

integrating by parts again wth $u = \cos(at)$ and $dv = e^{-st}$, we get

$$\begin{aligned}\mathcal{L}\{\sin(at)\} &= \frac{a}{s}\left[\frac{-1}{s}e^{-st}\cos(at) - \int_0^\infty \frac{a}{s}e^{-st}\sin(at)dt\right] \\ &= \frac{a}{s}\left[\frac{1}{s} - \frac{a}{s}\mathcal{L}\{\sin(at)\}\right]\end{aligned}$$

solving for $\mathcal{L}\{\sin(at)\}$, we get

$$\left(1 + \left(\frac{a}{s}\right)^2\right)\mathcal{L}\{\sin(at)\} = \frac{a}{s^2} \implies \mathcal{L}\{\sin(at)\} = \frac{a}{s^2 + a^2}$$

Quite interestingly, we can use the fact that $\sin(at) = \frac{1}{2i}\left(\exp(iat) - \exp(-iat)\right)$, and use linearity and the formula for $\mathcal{L}\{\exp(rt)\}$ to find the same result with considerably less pain. In the words of Jaques Hadamard, "the shortest way between two truths in the real domain passes through the complex field"