MAT 22B - Lecture Notes

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Nonhomogeneous Equations

We ended last time talking about the method of undetermined coefficients. This is a method to find a solution to an equation of the form

$$ay'' + by' + cy = g(t)$$

It is a guess-and-check method in which we guess the form of a solution based on the form of $g(t)$, but leave the coefficients undetermined. Then we plug in the guess and get a system of equations that the undetermined coefficients must satisfy, and solve this system to get the right numbers.

The method gets messy in two ways - first, in figuring out exactly what kind of solution to guess, and second, in doing the differentiation and algebra to solve for the coefficients. The second issue is not terribly deep, and if you ever get stuck, a computer can do most of the work for you. However, the first issue can be quite subtle, so I'd like to expand on the logic you should use in figuring out a good guess.

So far, we've seen cases where $g(t)$ is an exponential, a trig function, and a polynomial. The guess in each of these cases is pretty straightforward - just duplicate the same form as $g(t)$, but include both sine and cosine if $g(t)$ is either one.

A natural next step is to consider the case that $g(t)$ is a product of two of the basic kinds of functions we've considered so far. Let's consider an example

$$y'' - 3y' - 4y = -8e^t \cos(2t)$$

What would be a good guess in this case? Well, based on experience with the product rule, we know that a term like $e^t \cos(2t)$ will become e^t times a linear combination of $\cos(2t)$ and $\sin(2t)$ upon differentiation. So, we should make a guess of $Y = e^t(A\cos(2t) + B\sin(2t))$. Doing the computations we get

$$\begin{aligned} Y' &= e^t\left((-2A + B)\sin(2t) + (A + 2B)\cos(2t)\right) \\ Y'' &= e^t\left((-4A - 3B)\sin(2t) + (4B - 3A)\cos(2t)\right) \end{aligned}$$

Plugging these into the left-hand side of the ODE and collecting terms gives

$$\begin{aligned} Y'' - 3Y' - 4Y &= e^t\left((-4A - 3B + 6A - 3B - 4B)\sin(2t) + (4B - 3A - 3A - 6B - 4A)\cos(2t)\right) \\ &= e^t\left((2A - 10B)\sin(2t) + (-2B - 10A)\cos(2t)\right) \end{aligned}$$

Now we can set this equal to $-8e^t\cos(2t)$, and get the system of equations

$$\begin{aligned} 2A - 10B &= 0 \\ -10A - 2B &= -8 \end{aligned}$$

which we solve to get $A = \frac{10}{13}$ and $B = \frac{2}{13}$, so we've arrived at a solution

$$Y(t) = \frac{10}{13}e^t\cos(2t) + \frac{2}{13}e^t\sin(2t)$$

The same logic applies to products of eponentials with polynomials, trig functions with polynomials, or products of all three (though the algebra quickly begins to get quite nasty). The last real subtlety that occurs with undetermined coefficients is the case when your guess includes a solution to the corresponding homogeneous equation. Our example for this was

$$y'' - 3y' - 4y = e^{-t}$$

Guessing $Y = Ae^{-t}$ and trying to solve for A gives us $0A = 1$, which clearly has no solution. The fix I told you works is to just multiply by t, and guess $Y = Ate^{-t}$ instead. This seems pretty arbitrary at first, so I want to motivate it with a simpler example that we know how to solve analytically.

Consider the equation

$$y' + y = e^{-t}$$

This is a nonhomogeneous first order ODE with constant coefficients. If we try to apply the method of undetermined coefficients, we'd guess $Y = Ae^{-t}$, and again arrive at $0A = 1$. Fortunately, we can solve this equation using an integrating factor. The integrating factor here is $\exp(\int p(t)dt) = e^t$, so we get

$$\frac{d}{dt}\left(e^t y\right) = e^t e^{-t} = 1$$

Integrating, we get

$$e^t y = t + c \implies y = te^{-t} + ce^{-t}$$

where c is a constant of integration. Notice that the expression ce^{-t} is the general solution to the corresponding homogeneous equation, $y' + y = 0$, and that te^{-t} is a solution to the nonhomogeneous equation $y' + y = e^{-t}$. So, in this case, we see that it works to simply multiply by t.

To see that this is pretty sensible in general, consider solving the equation $ay'' + by' + cy = g$, where g is a solution to $ay'' + by' + cy = 0$, and let $Y = tg$. Then

$$\begin{aligned} aY'' + bY' + cY &= t(ag'' + bg' + cg) + 2ag' + bg \\ &= 2ag' + bg \end{aligned}$$

So heuristically, the t factor dropped out upon being plugged into the left-hand side of the equation, leaving something that can be comparable to g.

In any case, this is all to say that multiplying by t is, in this case, an effective "patch" to use when some part of your guess is a solution to the homogeneous equation, and leads to your system of undetermined coeffieicnets being unsolvable. I refer again to the table:

$g(t)$	Guess at $Y(t)$
$P_n(t) = a_n t^n + \cdots + a_1 t + a_0$	$t^s \left(A_n t^n + \cdots + A_1 t + A_0\right)$
$P_n(t)e^{at}$	$t^s \left(A_n t^n + \cdots + A_1 t + A_0\right) e^{at}$
$P_n(t)e^{at} \begin{cases} \sin bt \\ \cos bt \end{cases}$	$t^s e^{at} \left[(A_n t^n + \cdots + A_1 t + A_0)\cos bt + (B_n t^n + \cdots + B_1 t + B_0)\sin bt\right]$

where $P_n(t)$ denotes a polynomial of degree n, and s is the smallest integer (either $0, 1,$ or 2) that ensures no part of $Y(t)$ is a solution to the corresponding homogeneous equation.

Finally a word about sums, that I hope will be fairly intuitive. Consider the equation

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t$$

What in the world could we guess as a solution to this beast? Well, let's take a step back. If Y_1 is a solution to

$$y'' - 3y' - 4y = 3e^{2t}$$

and Y_2 is a solution to

$$y'' - 3y' - 4y = 2\sin t$$

and Y_3 is a solution to

$$y'' - 3y' - 4y = -8e^t \cos 2t$$

then $Y = Y_1 + Y_2 + Y_3$ is a solution to

$$y'' - 3y' - 4y = 3e^{2t} + 2\sin t - 8e^t \cos 2t$$

by our old, dependable friend, linearity.

So, to summarize, undetermined coefficients is a method that can deal with solving nonhomogeneous, second order ODEs with constant coefficients, whose nonhomogeneous term is anything that can be built from exponentials, trig functions, and polynomials by addition and/or multiplication.

Forced Oscillations

Here's where it gets interesting. To see what the physical meaning of a nonhomogeneous equation might be, let's think again about a mass on a spring, possibly with damping. The equation of motion is

$$F = ma$$

where $a = \frac{d^2y}{dt^2}$ is acceleration, and there is a damping force $-b\frac{dy}{dt}$ and a spring force $-ky$. In the past we stopped here, and considered what happened when we kick the mass (i.e. set initial conditions) and let it go. Now, though, we have the tools to describe what happens when we can actively push the mass in a time-dependent way. If we denote by $F_{ext}(t)$ the external force on the mass at time t, our ODE becomes

$$m\frac{d^2y}{dt^2} = -b\frac{dy}{dt} - ky + F_{ext}(t)$$

or

$$m\frac{d^2y}{dt^2} + b\frac{dy}{dt} + ky = F_{ext}(t)$$

so the motion $y(t)$ is exactly a solution to a nonhomogeneous, linear, second order ODE with constant coefficients! LET'S GO.

The most interesting phenomenon that can happen here is called *resonance* (e.g. Galloping Gertie). This is how swings work. The basic idea is that when you apply a periodic force to an oscillating thing at the same frequency at which it would naturally oscillate, the oscillations steadily grow in amplitude. A hugely important example is called aerodynamic flutter, in which resonance happens in airplane wings. Aerodynamic flutter is a very bad thing. There is, in fact, a lot of work done in mechanical design to control the natural vibration frequencies of designed objects so that they resist being resonantly driven.

Let's see what that means. To start, we'll consider no damping, and choose a mass $m = 1$ and spring constant $k = 1$. If the mass were not driven, its motion $y(t)$ would obey the ODE

$$y'' + y = 0$$

So clearly, $y = c_1 \cos(t) + c_2 \sin(t)$. Now, what if we drive it with a sinusoidal force? That is, set $F_{ext}(t) = \sin(t)$. Then we have

$$y'' + y = \sin(t)$$

Let's look for a solution using the method of undetermined coefficients. As we saw, we can't use $Y = A\sin(t) + B\cos(t)$ because we'd get $0A = 1 = 0B$. So let's try $Y = t(A\sin(t) + B\cos(t))$. Then

$$\begin{aligned} Y' &= t(A\cos t - B\sin t) + A\sin t + B\cos t \\ &= (At + B)\cos t + (A - Bt)\sin t \\ Y'' &= A\cos t - (At + B)\sin t - B\sin t + (A - Bt)\cos t \\ &= (2A - Bt)\cos t + (-2B - At)\sin t \end{aligned}$$

so

$$Y'' + Y = -2B\sin t + 2A\cos t$$

setting this equal to $\sin t$ gives $B = \frac{-1}{2}$ and $A = 0$, so our solution is $Y = \frac{-1}{2}t\cos t$. Notice that the amplitude of this function grows without bound as $t \to \infty$! Resonance! More on this next time.

Variation of Parameters

Variation of Parameters is another method to find a solution to a nonhomogeneous equation. It's very different in flavor to undetermined coefficients, in that it's completely deductive - very little is guesswork. The only drawback is that we need to begin with a fundamental set of solutions to the corresponding homogeneous equation.

The way it works is this. We have an ODE

$$y'' + p(t)y' + q(t)y = g(t)$$

which is second order, linear, and nonhomogeneous. Say $\{y_1, y_2\}$ is a fundamental set of solutions of the CHE

$$y'' + p(t)y' + q(t)y = 0$$

So for any c_1, c_2, we have $y = c_1y_1 + c_2y_2$ solves the CHE. Now we'll pull a trick similar to what happened in "reduction of order". We guess that instead of taking c_1 and c_2 to be constant, we let them depend on time. That is, we consider a function

$$y(t) = u_1(t)y_1(t) + u_2(t)y_2(t)$$

and see if there are choices we can make for u_1 and u_2 to make y a solution of the original, nonhomogeneous equation. Now before we proceed, I want to point out that in the end, we will end up with some equation relating u_1 and u_2 to y_1, y_2, p, q, and g. This will be a single equation for the two unknown functions u_1, u_2. Heuristically, that means we should be able to impose one more condition on this pair of functions and still expect there could be a solution.

So, let's plug it into the left-hand side and see what we get. We have

$$y' = u_1y_1' + u_1'y_1 + u_2y_2' + u_2'y_2$$

It is at this point that we'll impose our extra condition, in a way that will make the calculation much easier. We will require that

$$u_1'y_1 + u_2'y_2 = 0$$

so that now

$$y' = u_1y_1' + u_2y_2'$$

One very nice feature that this will give us is that y' now involves no derivatives of the u's, and so y'' will involve only first derivatives of the u's. Taking the next derivative, we get

$$y'' = u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2'$$

and putting this together gives

$$\begin{aligned} y'' + p(t)y' + q(t)y &= u_1y_1'' + u_1'y_1' + u_2y_2'' + u_2'y_2' + p\left(u_1y_1' + u_2y_2'\right) + q\left(u_1y_1 + u_2y_2\right) \\ &= u_1\left(y_1'' + py_1' + qy_1\right) + u_2\left(y_2'' + py_2' + qy_2\right) + u_1'y_1' + u_2'y_2' \\ &= u_1'y_1' + u_2'y_2' \end{aligned}$$

where we've used that y_1 and y_2 each is a solution to the CHE. So our condition on u_1, u_2 is now that $u_1' y_1' + u_2' y_2' = g$. Notice now that we can write this along with our first constraint as a system of equations

$$
\begin{aligned}
u_1' y_1 + u_2' y_2 &= 0 \\
u_1' y_1' + u_1' y_2 &= g
\end{aligned}
$$

Note that this is simply a system of *algebraic equations* for the pair of functions u_1', u_2'. We can write it in matrix form as

$$
\begin{bmatrix} y_1 & y_2 \\ y_1' & y_2' \end{bmatrix} \begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \begin{bmatrix} 0 \\ g \end{bmatrix}
$$

Remember the Wronskian? Me neither. Inverting the matrix above we get

$$
\begin{bmatrix} u_1' \\ u_2' \end{bmatrix} = \frac{1}{y_1 y_2' - y_2 y_1'} \begin{bmatrix} y_2' & -y_2 \\ -y_1' & y_1 \end{bmatrix} \begin{bmatrix} 0 \\ g \end{bmatrix}
$$

or, denoting $W(y_1, y_2)(t) = y_1(t) y_2'(t) - y_2(t) y_1'(t)$, we get

$$
u_1'(t) = \frac{-y_2(t) g(t)}{W(y_1, y_2)(t)} \quad \text{and} \quad u_2'(t) = \frac{y_1(t) g(t)}{W(y_1, y_2)(t)}
$$

So integrating we have

$$
u_1 = \int \frac{-y_2(t) g(t)}{W(y_1, y_2)(t)} dt \quad \text{and} \quad u_2 = \int \frac{y_1(t) g(t)}{W(y_1, y_2)(t)} dt
$$

Notice that each integral comes with a constant of integration. Including these constants in the expression $u_1 y_1 + u_2 y_2$ will give us a term $c_1 y_1 + c_2 y_2$, which is the general solution of the CHE.