# MAT 22B - Lecture Notes

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## Nonhomogeneous Equations

Now that we've seen much of the theory of solving homogeneous second order linear ODE's, we're ready to think about solving nonhomogeneous ones. That is, we'll be thinking about equations of the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where now $g(t) \neq 0$. We will sometimes call $g(t)$ the "nonhomogeneous term".

To lay the foundation, I again want to emphasize the parallels with linear algebra. As we did before, we'll denote the left-hand side of the ODE by

$$L[y] := y'' + py' + qy$$

so that the equation we're studying is $L[y] = g$. Our goal will be to find the *general solution*, meaning an expression which describes every possible function $y$ that satisfies $L[y] = g$.

In the homogeneous case, $L[y] = 0$, we rediscovered a fact from linear algebra, which says that the set of all solutions is a vector space. This fact follows from the fact that the left-hand side $L$ is linear, and is proved in exactly the same way as showing that the nullspace of a matrix is a vector space.

As I alluded to before, solving $L[y] = g$ is just like solving $Ax = b$. To see what I mean, consider the following fact: If $Y_1$ and $Y_2$ are two solutions to $L[y] = g$, then their difference is a solution to the *corresponding homogeneous equation* (CHE), $L[y] = 0$. To see why this is true, we can just compute $L[Y_1 - Y_2]$, remembering that $L$ is linear, and get

$$\begin{aligned} L[Y_1 - Y_2] &= L[Y_1] - L[Y_2] \\ &= g - g = 0 \end{aligned}$$

This statement is quite straightforward, but its implications are huge. What this gives us is that if I somehow find a *single* solution $Y$ to the nonhomogeneous equation $L[y] = g$, then I can get *any other* solution by adding a function which satisfies the CHE. The upshot is that if $\{y_1, y_2\}$ is a fundamental set of solutions to the CHE $L[y] = 0$, then the general solution to $L[y] = g$ is

$$Y + c_1 y_1 + c_2 y_2$$

In this way, the tools we developed for finding fundamental sets of solutions to homogeneous equations continue to be critically useful in our study of nonhomogeneous equations. The only thing we're missing is a way to find a *single* solution to $L[y] = g$. The first method we will see to do this is called the method of undetermined coefficients

## Undetermined Coefficients

This method is yet another guess-and-check method, similar to the reasoning that got us to the characteristic equation and the method of reduction of order. Before we go into details, I should point out that this method is only applicable to the constant coefficient case - that is, ODEs of the form $ay'' + by' + cy = g(t)$.

The basic idea is to look at the nonhomogeneous term $g(t)$ and guess that there's a solution that is of a similar form. It's best to illustrate this by example:

### An exponential function

Consider the nonhomogeneous ODE

$$y'' - 3y' - 4y = 3e^{2t}$$

The nonhomogeneous term, $3e^{2t}$, is an exponential. Because we know that exponential functions come back to themselves after differentiation, we might reasonably guess that the solution also looks like $e^{2t}$. So we'll make a guess that $Y = Ae^{2t}$ is a solution for some constant $A$. Let's plug this into the ODE and see what we get:

$$\begin{aligned} Y'' - 3Y' - 4Y &= 4Ae^{2t} - 6Ae^{2t} - 4Ae^{2t} \\ &= -6Ae^{2t} \end{aligned}$$

and we need this expression to be equal to $3e^{2t}$ if $Y$ is going to be a solution. That means that $-6A = 3$, so $A = -\frac{1}{2}$, and we have that $Y = \frac{-1}{2}e^{2t}$ is a solution to the ODE. If you don't believe me, check it yourself!

We can now see why this method is called "undetermined coefficients". We made a guess that had a coefficient in it, and that coefficient was left undetermined. Plugging it in to the ODE told us what it had to be, and we end up with a solution!

### A trig function

Let's move on to a different type of nonhomogeneous term - a trig function. Let's find a solution to the ODE

$$y'' - 3y' - 4y = 2\sin(t)$$

Based on our reasoning before, we might guess that $Y = A\sin(t)$ will work. So we plug in our guess and see what happens:

$$\begin{aligned} Y'' - 3Y' - 4Y &= -A\sin(t) - 3A\cos(t) - 4A\sin(t) \\ &= -5A\sin(t) - 3A\cos(t) \end{aligned}$$

Well, this sucks. No matter what $A$ we choose, there's no way to get rid of that pesky cosine term! Good grief. I guess this means it's time to give up and go home.

But soft! What light through yonder window breaks? It is the east, and $\cos(t)$ is the sun. Let's see what we get with a guess that's some combination of $\cos(t)$ and $\sin(t)$. That is, let $Y = A\sin(t) + B\cos(t)$. We get

$$\begin{aligned} Y'' - 3Y' - 4Y &= -5A\sin(t) - 3A\cos(t) - B\cos(t) + 3B\sin(t) - 4B\cos(t) \\ &= (-5A + 3B)\sin(t) + (-3A - 5B)\cos(t) \end{aligned}$$

Setting this expression equal to $2\sin(t)$ gives us two equations for the two unknowns, $A$ and $B$. We have

$$\begin{aligned} -3A - 5B &= 0 \\ -5A + 3B &= 2 \end{aligned}$$

so $A = -5/17$ and $B = 3/17$, and the solution is

$$Y(t) = \frac{-5}{17}\sin(t) + \frac{3}{17}\cos(t)$$

Fractions suck but we're all grownups. More importantly, computers exist.

### A polynomial

It turns out we can even apply this method to equations whose nonhomogeneous term is a polynomial. For instance,

$$y'' - 3y' - 4y = 4t^2 - 1$$

The intuition we've been working up towards would tell us to guess that there's a solution which is a polynomial. This is the right thing to do. Given that differentiation decreases the order of polynomials, we need to start with one of at least second order (to be able to get the $4t^2$ on the right-hand side). So we'll guess $Y = At^2 + Bt + C$. Notice there are now 3 undetermined coefficents, and we'll get 3 equations for them at the end by equating coefficients of each power of $t$. Let's do it!

$$\begin{aligned} Y'' - 3Y' - 4Y &= 2A - 3(2At + B) - 4(At^2 + Bt + C) \\ &= -4At^2 + (-6A - B)t + (2A - 3B - 4C) \end{aligned}$$

Equating this to $4t^2 - 1$ gives, first, $A = -1$, then $B = 6$, then $C = -5$. So our solution is $Y(t) = -t^2 + 6t - 5$.

### Combinations

We can play a similar game for functions that are *products* of the types of functions we just dealt with. For instance, if the nonhomogeneous term is $e^{2t}\cos(3t)$, we should guess a solution $Y(t) =$

$e^{2t}(A\cos(3t) + B\sin(3t))$, becuase differentiating will turn sine and cosine into each other, while the factor $e^{2t}$ will stay multiplying each term.

There's one more strange situation that can arise - again, let's look at an example.

Consider the ODE

$$y'' - 3y' - 4y = e^{-t}$$

According to what we did before, we can guess that $Y = Ae^{-t}$ is a solution. Plugging it in we get

$$Y'' - 3Y' - 4Y = e^{-t} + 3e^{-t} - 4e^{-t} = 0$$

which can never equal $e^{-t}$, no matter what $A$ we pick! The issue is clear: our guess was a solution to the corresponding homogeneous equation. In this case, we should modify our guess by multiplying by $t$. That is, guess $Y = Ate^{-t}$. In this case, $Y' = A(e^{-t} - te^{-t})$ and $Y'' = A(-2e^{-t} + te^{-t})$, so we get

$$\begin{aligned} Y'' - 3Y' - 4Y &= e^{-t}\left(A\left(-2+t\right) - 3A(1-t) - 4At\right) \\ &= e^{-t}\left(-5A\right) + te^{-t}\left(A + 3A - 4A\right) \end{aligned}$$

so $A = -1/5$, and the solution is $Y = \frac{-1}{5}te^{-t}$.

So that's basically it. The idea behind undetermined coefficients is fairly straightforward. The trickiest parts are:

1. Knowing the right form to guess
2. Doing the algebra to solve for all of the coefficients

There is a table in your book (table 3.5.1 on page 182) that covers the first point. It's recreated here ($P_n$ denotes some polynomial of degree $n$, and $s = 0, 1, 2$ is the smallest whole number such that your guess is not a solution to the corresponding homogeneous equation)

| $g(t)$ | Guess at $Y(t)$ |
|---|---|
| $P_n(t) = a_nt^n + \cdots + a_1t + a_0$ | $t^s\left(A_nt^n + \cdots + A_1t + A_0\right)$ |
| $P_n(t)e^{at}$ | $t^s\left(A_nt^n + \cdots + A_1t + A_0\right)e^{at}$ |
| $P_n(t)e^{at}\begin{cases}\sin bt \\ \cos bt\end{cases}$ | $t^se^{at}\left[\left(A_nt^n + \cdots + A_1t + A_0\right)\cos bt + \left(B_nt^n + \cdots + B_1t + B_0\right)\sin bt\right]$ |

The algebra always just sucks a bit. Computers help.