MAT 22B - Lecture Notes

19 August 2015

Repeated Roots/Reduction of Order

We've tackled *almost* everything that can happen in solving an second order, linear, homogeneous ODE with constant coefficients. That is, one of the form

$$ay'' + by' + cy = 0$$

where a, b, c are constants. To find solutions we guess that $y = e^{rt}$ is a solution for some r, then derive conditions on r that make that true. It turns out that $ar^2 + br + c = 0$ is sufficient. This equation is called the *characterisitc equation*, and its roots, given by

$$r_{\pm} = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

give us solutions, $y_1 = \exp(r_+ t)$ and $y_2 = \exp(r_- t)$. This is all well and good if $r_{\pm}$ are real-valued, but for complex $r_{\pm}$, the functions $\exp(r_{\pm} t)$ are complex-valued, and this won't do if $y(t)$ is meant to be some physical quantity. So, we saw that Euler's formula comes to the rescue, and we find that if $r_{\pm} = \lambda \pm i\mu$, then we have a pair of real-valued solutions $y_1 = \exp(\lambda t)\cos(\mu t)$ and $y_2 = \exp(\lambda t)\sin(\mu t)$.

What we need to make sure of, though, is that this pair of functions generates *every* possible solution to the ODE. As we've mentioned, the set of solutions to $ay'' + by' + cy = 0$ forms a vector space of dimension two, which means that we can write it as the span of a basis that consists of two functions. Checking whether or not any given pair of functions will work as a basis is the job of the Wronskian:

$$W(y_1, y_2)(t_0) = \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

which is nonzero exactly when $\{y_1, y_2\}$ forms a basis for the solution space. If they do form a basis, we call the set $\{y_1, y_2\}$ a fundamental set of solutions.

It turns out (do the calculation) that when $r_+ \neq r_-$, the solutions described above do indeed form a fundamental set of solutions (both in the real and the complex case). The last little bit of trouble could arise if $r_+ = r_- := r$, and our "recipe" only tells us one solution: $y_1 = \exp(rt)$. Clearly, we need a y_2 from somewhere to form a fundamental set of solutions. But how do we find it?

The key insight is this: we know, by linearity, that for any constant k, the function $y = ke^{rt}$ is also a solution. But what if we multiplied by something that was *not* constant? That is, can we find a (non-constant) function $v(t)$ such that $v(t)e^{rt}$ is also a solution to the ODE?

Well, let's see! We can plug the expression $y = v(t)e^{rt}$ in to the ODE and see what conditions on v have to hold in order for the ODE to be satisfied. We'll need to compute y' and y'', which are

$$\begin{aligned} y' &= v(t)re^{rt} + v'(t)e^{rt} = e^{rt}\big(rv(t) + v'(t)\big) \\ y'' &= e^{rt}\big(rv'(t) + v''(t)\big) + re^{rt}\big(rv(t) + v'(t)\big) = e^{rt}\big(v''(t) + 2rv'(t) + r^2v(t)\big) \end{aligned}$$

inserting these into the left-hand side of the ODE we get

$$\begin{aligned} ay'' + by' + cy &= ae^{rt}\big(v''(t) + 2rv'(t) + r^2v(t)\big) + be^{rt}\big(rv(t) + v'(t)\big) + ce^{rt}v(t) \\ &= e^{rt}\big(av''(t) + (2ra + b)\,v'(t) + \big(ar^2 + br + c\big)\,v(t)\big) \end{aligned}$$

Now, we use what we know about r. In particular, we know it's a root of the characteristic equation - that is, $ar^2 + br + c = 0$, and the $v(t)$ term above vanishes. Also, we know (by the quadratic formula) that $r = \frac{-b}{2a}$, so the coefficient of $v'(t)$ also vanishes. So we're left with

$$ay'' + by' + cy = ae^{rt}v''(t)$$

and the above function has to be zero in order for $v(t)e^{rt}$ to be a solution. Since $a \neq 0$ and $e^{rt} \neq 0$, we must have $v''(t) = 0$. In other words, $v(t) = c_1 + c_2t$, for any constants c_1, c_2 will work, and the function

$$y = (c_1 + c_2t)e^{rt} = c_1e^{rt} + c_2te^{rt}$$

is a solution. In fact, this expression *is* the general solution, and the pair of functions $\{e^{rt}, te^{rt}\}$ is a basis, or fundamental set of solutions.

So in summary, the general solution of the ODE $ay'' + by' + cy = 0$ falls into one of three cases:

1. The roots $r_\pm$ of $ar^2 + br + c = 0$ are real and distinct. Then the general solution is $c_1 \exp(r_+t) + c_2 \exp(r_-t)$
2. The roots $r_\pm$ of $ar^2 + br + c = 0$ are complex conjugates of each other, $r_\pm = \lambda \pm i\mu$. Then the general solution is $c_1 \exp(\lambda t)\cos(\mu t) + c_2 \exp(\lambda t)\sin(\mu t)$
3. The discriminant $b^2 - 4ac = 0$, and there is only one root r of $ar^2 + br + c = 0$. Then the general solution is $c_1 \exp(rt) + c_2t\exp(rt)$

And that's it.

"Reduction of Order"

It's worth thinking for a moment about what exactly it was that we did which was a "reduction of order". This term is applied to the trick we pulled in looking for a solution of the form $v(t)e^{rt}$. We ended up with the condition that $v'' = 0$, which is still a second order ODE... but I can also write it as a first order ODE, $w' = 0$, where I've substituted $w = v'$. This sounds trivial and stupid, but let me explain.

The trick of guessing $v(t)y_1(t)$ as a solution when you already know one solution y_1 will work for any second order, linear, homogeneous ODE. That is, if y_1 is a solution to the ODE

$$y'' + p(t)y' + q(t)y = 0$$

Then I can plug in $y = v(t)y_1(t)$ and see if there are conditions I can impose on v that will make this y satisfy the ODE. As before, we can write out y' and y'', to get

$$\begin{aligned} y' &= v'y_1 + vy_1 \\ y'' &= v''y_1 + v'y_1' + v'y_1 + vy_1'' \end{aligned}$$

So the left-hand side of the ODE becomes

$$\begin{aligned} y'' + py' + qy &= v''y_1 + v'(y_1' + y_1) + vy_1'' + p\left(v'y_1 + vy_1\right) + qvy_1 \\ &= y_1v'' + \left(2y_1' + py_1\right)v' + \left(y_1'' + py_1 + qy_1\right)v \end{aligned}$$

Now, the fact that y_1 solves the ODE means that the coefficient of v above is equal to zero! This is the *same thing* that we saw happen for the constant coefficient case, when the characteristic polynomial showed up multiplying v. So now the ODE reads

$$y_1v'' + \left(2y_1' + py_1\right)v' = 0$$

and now it's a bit more clear why pulled the strange trick of defining $w = v'$. If I do this to the above equation, I get

$$y_1w' + \left(2y_1' + py_1\right)w = 0$$

which is a first order equation for the function $w(t)$, which we can solve in a moment by separating variables. Note that whenever we find a function $w(t)$ that satisfies the above first order equation and integrate it to obtain $v(t)$, we are free to add a constant of integration: That is,

$$y = (c_1 + v(t))y_1$$

is a solution for any c_1. In this sense, the method of reduction of order gives us back the fact that a constant multiple of y_1 is still a solution, which we already knew. So that's kind of nice I guess.

Example

This is problem 24 from section 3.4 in the text.

Given that $y_1(t) = t$ is a solution to the ODE $t^2y'' + 2ty' - 2y = 0$, $t > 0$, use the method of reduction of order to find a second solution.

The method of reduction of order says that we should suppose there's some $v(t)$ such that $y = v(t)y_1(t)$ is also a solution, and then go and find what the $v(t)$ is. In our case, the new solution should look like $y = tv(t)$. So let's compute its derivatives to plug it into the equation.

$$\begin{aligned} y' &= tv' + v \\ y'' &= tv'' + 2v' \end{aligned}$$

Then the left-hand side of the ODE is

$$\begin{aligned} t^2y'' + 2ty' - 2y &= t^2\left(tv'' + 2v'\right) + 2t\left(tv' + v\right) - 2tv \\ &= t^3v'' + 4t^2v' + (2t - 2t)\,v \\ &= t^3v'' + 4t^2v' \end{aligned}$$

If this expression is equal to zero, then we have

$$t^3v'' + 4t^2v' = 0$$

Letting $w = v'$, we have

$$t^3w' + 4t^2w = 0$$

separating variables:

$$\frac{w'}{w} = \frac{-4t^2}{t^3} = \frac{-4}{t}$$

integrating:

$$\ln(w) = -4\ln(t) + k$$

exponentiating:

$$w(t) = e^k e^{-4\ln(t)} = e^k t^{-4}$$

Then integrating to get $v(t)$

$$v(t) = \int w(t)dt = \frac{-1}{3}e^k t^{-3} + c_1$$

So the general solution is

$$\begin{aligned} y &= tv(t) = t\left(\frac{-1}{3}e^k t^{-3} + c_1\right) \\ &= c_1t + c_2t^{-2} \end{aligned}$$

where $c_2 = \frac{-1}{3}e^k$. So the new solution we've found is the second term, $y_2 := t^{-2}$. As an exercise, go back and double check that y_2 does, in fact, satisfy the ODE, and that $W(y_1, y_2)(t_0) \neq 0$ for any $t > 0$.

Aside: Euler Equations

Euler equations are ODEs of the form

$$t^2\frac{d^2y}{dt^2} + \alpha t\frac{dy}{dt} + \beta y = 0, \quad t > 0$$

Notice that the coefficients t^2 and αt are not constant, so the characteristic equation method does not apply. However, we can solve such equations using a substitution that turns them into constant coefficient ODEs. That is, define

$$x = \ln t$$

and use x, rather than t, as the independent variable. By the chain rule

$$\frac{dy}{dt} = \frac{dy}{dx}\frac{dx}{dt} = \frac{dy}{dx}\frac{1}{t}$$

and

$$\frac{d^2y}{dt^2} = \frac{d}{dt}\left(\frac{dy}{dx}\frac{dx}{dt}\right) = \frac{d^2y}{dxdt}\frac{dx}{dt} + \frac{dy}{dx}\frac{d^2x}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{dx}{dt}\right)^2 + \frac{dy}{dx}\frac{d^2x}{dt^2}$$

which comes out to

$$\frac{d^2y}{dt^2} = \frac{d^2y}{dx^2}\left(\frac{1}{t}\right)^2 + \frac{dy}{dx}\left(\frac{-1}{t^2}\right)$$

Inserting these into the ODE gives

$$t^2\left(\frac{d^2y}{dx^2}\left(\frac{1}{t}\right)^2 + \frac{dy}{dx}\left(\frac{-1}{t^2}\right)\right) + \alpha t\left(\frac{dy}{dx}\frac{1}{t}\right) + \beta y = \frac{d^2y}{dx^2} + (\alpha - 1)\frac{dy}{dx} + \beta y = 0$$

which is now a constant coefficient ODE for $y(x)$! We can now solve it by the characteristic equation, whose roots are

$$r_\pm = \frac{1 - \alpha \pm \sqrt{(\alpha-1)^2 - 4\beta}}{2}$$

so $y = \exp(r_\pm x)$. Now, transforming back into t-land, we get

$$\begin{aligned} y(t) &= y(x(t)) \\ &= \exp(r_\pm \ln(t)) = t^{r_\pm} \end{aligned}$$

So Euler equations have solutions that are just powers of t! If we look back at the equation, it makes sense - differentiating t^r gives us t^{r-1}, and the factor of t in front of y' makes the term comparable to y itself, and the same for t^2y''. Also notice that our example of reduction of order used an Euler equation.