MAT 22B - Lecture Notes

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Wrapping up Mass on Spring with Damping

The ODE we derived from physical principles ($F = ma$) was

$$my'' + by' + ky = 0$$

If we consider the case of zero damping, i.e. $b = 0$, the characteristic equation has roots $r_{\pm} = \pm\sqrt{-\frac{k}{m}} = \pm i\sqrt{\frac{k}{m}}$. Hence, the (real-valued) solutions are $y_1 = \cos\left(t\sqrt{\frac{k}{m}}\right)$ and $y_2 = \sin\left(t\sqrt{\frac{k}{m}}\right)$. Note that this makes sense if we think about it physically: increasing the mass m will make it oscillate more slowly, and a stiffer spring (i.e. larger k) will make it oscillate more quickly.

If we include damping, the roots of the characteristic equation are

$$r_{\pm} = \frac{-b \pm \sqrt{b^2 - 4mk}}{2m} = \frac{-b}{2m} \pm \sqrt{\frac{b^2 - 4mk}{4m^2}}$$

So, depending on the parameter values, the roots could be real and distinct, repeated, or complex. Let's focus on the complex case first - that is, $4mk > b^2$. In this case, the real-valued solutions are

$$\begin{aligned} y_1 &= \exp\left(-\frac{b}{2m}t\right)\cos(\omega t) \\ y_2 &= \exp\left(-\frac{b}{2m}t\right)\sin(\omega t) \end{aligned}$$

where $\omega = \sqrt{\frac{k}{m} - \left(\frac{b}{2m}\right)^2}$. Notice two things: the frequency ω has decreased as compared to the un-damped case, which makes physical sense, and the magnitude of each of the solutions decays exponentially due to the $\exp\left(-\frac{b}{2m}t\right)$ term. The rate of this decay increases with increased damping (b), and decreases with increasing m. That is, a larger mass has more inertia.

We can also consider the case of real, distinct roots. This happens if the damping constant b is too large, and so we typically say that the oscillator is "over-damped". In this case, you can convince yourself (both physically and mathematically) that both of these roots are negative. The solutions aren't very interesting; they simply decay to zero, with possibly a single change in direction. You can visualize a mass that is damped so strongly that it doesn't have time to actually oscillate at all. The case of repeated roots is qualitatively similar to the over-damped case, but we don't yet have the tools to write down the solution

Solutions of Linear Homogeneous Equations

My intention with this lecture is to emphasize connections to linear algebra.

Last time, we saw how to solve equations of the form

$$ay'' + by' + cy = 0$$

using the characteristic equation to find numbers r such that e^{rt} solves the DE. We saw that to get the general solution to such an equation, we need two different solutions, $e^{r_1 t}$ and $e^{r_2 t}$ (as long as $b^2 \neq 4ac$), and then every solution has the form $c_1 e^{r_1 t} + c_2 e^{r_2 t}$. This can be read as a statement about the structure of the set of solutions to the ODE: in particular, the set is a two-dimensional vector space (with basis $\{e^{r_1 t}, e^{r_2 t}\}$). We'll now talk a little bit more generally about the sets of solutions of second order, linear, homogeneous equations.

A second order linear ODE has the form

$$a_2(t)y'' + a_1(t)y' + a_0(t)y = g(t)$$

but we will typically consider the form

$$y'' + p(t)y' + q(t)y = g(t)$$

where we have "divided through by $a_2(t)$", similarly to how we dealt with first order linear ODE's. Again, such an equation is called *homogeneous* if $g(t) \equiv 0$.

At this point, a piece of notation will help clarify our discussion considerably. Let's define an operator (i.e. machine, gadget, black box,...) called L, which eats functions and spits out other functions. We'll define it by saying that if you feed it a (twice-differentiable) function $\phi = \phi(t)$, it spits out

$$L[\phi] = \phi'' + p\phi' + q\phi$$

The square brackets are intended to emphasize that the input to L is a *whole function*, not just a number. Likewise, the right-hand side is a function of t. Its value at any given t is

$$L[\phi](t) = \phi''(t) + p(t)\phi'(t) + q(t)\phi(t)$$

If you want to impress your fancy mathematician friends and a cocktail party, you could write

$$L\colon C^2(\mathbb{R}) \to C(\mathbb{R})$$

meaning that L takes as input a twice-differentiable (C^2) function on $\mathbb{R}$, and gives as output a continuous (C) function on $\mathbb{R}$.

The point is that we can now write our differential equation in the compact form

$$L[y] = g$$

or, if the equation is homogeneous,

$$L[y] = 0$$

The most important feature of L is that it is a *linear operator*. This means that for any constants c_1, c_2, and any functions y_1, y_2, we have

$$L[c_1 y_1 + c_2 y_2] = c_1 L[y_1] + c_2 L[y_2]$$

To see this, we can just compute:

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= (c_1 y_1 + c_2 y_2)'' + p\,(c_1 y_1 + c_2 y_2)' + q\,(c_1 y_1 + c_2 y_2) \\ &= c_1 \left(y_1'' + p y_1' + q y_1\right) + c_2 \left(y_2'' + p y_2' + q y_2\right) \\ &= c_1 L[y_1] + c_2 L[y_2] \end{aligned}$$

Notice that this property is shared by matrix multiplication: if A is an $m \times n$ matrix, and $\vec{x}_1$ and $\vec{x}_2$ are n-dimensional vectors, then for any constants c_1, c_2, we have

$$A(c_1 \vec{x}_1 + c_2 \vec{x}_2) = c_1 A \vec{x}_1 + c_2 A \vec{x}_2$$

So, in a very literal sense, solving

$$L[y] = g$$

is just like solving

$$A\vec{x} = \vec{b}$$

and in particular, solving

$$L[y] = 0$$

is just like solving

$$A\vec{x} = \vec{0}$$

They key property of the set of solutions to $A\vec{x} = \vec{0}$ is that they form a *vector space*, usually called the *nullspace* of A. Likewise, solutions to $L[y] = 0$ form a vector space, since for any constants c_1, c_2 and any solutions y_1, y_2, we have

$$\begin{aligned} L[c_1 y_1 + c_2 y_2] &= c_1 L[y_1] + c_2 L[y_2] \\ &= c_1 \cdot 0 + c_2 \cdot 0 = 0 \end{aligned}$$

Imposing Initial Conditions

It should be noted that there is an existence and uniqueness theorem for ODE's of this form, and it is almost identical to the one for first-order linear ODE's. It says that if p, q, and g are all continuous on some interval $I \subseteq \mathbb{R}$ containing t_0, then the IVP

$$y'' + p(t)y' + q(t)y = g(t), \quad y(t_0) = y_0, \ y'(t_0) = y_0'$$

has a unique solution $y(t)$, which is defined and twice differentiable everywhere on I.

An important thing to note about this theorem is that it tells us existence and uniqueness. That means that for *any* pair of numbers (y_0, y_0'), we can stick them into the above IVP, and get *exactly one* solution that meets those initial conditions. Likewise, for any function $y(t)$ that satisfies the ODE, we can calculate $y(t_0)$ and $y'(t_0)$ and get the initial conditions it satisfies. This is all to say that an expression can rightly be called a *general solution* to the ODE if and only if it is in fact a solution, and can meet any initial conditions y_0, y_0' we want to impose.

So let's see how this looks in our special case of $g = 0$ - that is, for homogeneous equations. The special treat that we get in this case is that the set of solutions is a vector space, and as such, it has a *basis*. Moreover, because the ODE is second order, the solution space will be two dimensional (a fact that I haven't actually proven, but is true). Hence, there are functions y_1, y_2 so that *every* solution is of the form $c_1y_1 + c_2y_2$. We sometimes call this expression the *span* of $\{y_1, y_2\}$, just like in linear algebra.

Now, as we just established, to check that the expression $c_1y_1 + c_2y_2$ is the general solution, we should check that it is a solution and that it can meet any initial conditions. The first point is true as long as y_1 and y_2 are both solutions, by linearity and homogeneity of $L[y] = 0$. So let's see if we can pick c_1 and c_2 to meet an arbitrary pair of initial conditions. If y_0 and y_0' are two real numbers, then imposing them as initial conditions looks like

$$\begin{aligned} c_1y_1(t_0) + c_2y_2(t_0) &= y_0 \\ c_1y_1'(t_0) + c_2y_2'(t_0) &= y_0' \end{aligned}$$

or, in matrix form

$$\begin{bmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{bmatrix} \begin{bmatrix} c_1 \\ c_2 \end{bmatrix} = \begin{bmatrix} y_0 \\ y_0' \end{bmatrix}$$

So finding the constants c_1, c_2 has been reduced to inverting the matrix above. Now, recall that a matrix is invertible if and only if its *determinant is nonzero.* So if the determinant

$$W(y_1, y_2)(t_0) := \begin{vmatrix} y_1(t_0) & y_2(t_0) \\ y_1'(t_0) & y_2'(t_0) \end{vmatrix}$$

is not equal to zero, then the pair of functions $\{y_1, y_2\}$ spans the set of all solutions. In other words, $c_1y_1 + c_2y_2$ is the general solution, and we call the set $\{y_1, y_2\}$ a *fundamental set of solutions.* The above determinant is called the *Wronskian determinant*, or just the Wronskian.

You may be wondering at this point about the appearance of t_0 above. What if $W(y_1, y_2)(t_0)$ is zero for some t_0 and nonzero for others? Well, that simply doesn't happen if we stay within an interval I

on which existence and uniqueness of solutions holds. We can see this through a theorem called Abel's Theorem. It states that if y_1 and y_2 are both solutions to $y'' + py' + qy = 0$, then their Wronskian is given by

$$W(y_1, y_2)(t) = c \exp\left[-\int p(t)dt\right]$$

so that either $W = 0$ for all $t \in I$ (if $c = 0$), or $W \neq 0$ for all $t \in I$, since the exponential of anything is never zero. The point here is that when you compute the Wronskian, you can evaluate it on any point inside an interval I on which p, q, and g are continuous.

Another Perspective

The thing that the Wronskian is checking is whether or not y_1 and y_2 are linearly independent. Linear independence here means the same thing as it does in linear algebra: that the only c_1 and c_2 that satisfy $c_1y_1 + c_2y_2 = 0$ are $c_1 = c_2 = 0$. This statement is kind of hard to get your hands on, so let's think instead about linear *dependence*. The functions y_1, y_2 are linearly *dependent* if there *are* nonzero c_1, c_2 so that $c_1y_1 + c_2y_2 = 0$. We can now rearrange this to get

$$\frac{y_1}{y_2} = -\frac{c_2}{c_1}$$

Huh. So, if the functions are linearly dependent, their quotient is a constant. One way to check if something is a constant is to see if its derivative is zero. So we can check

$$\frac{d}{dt}\left[\frac{y_1}{y_2}\right] = \frac{y_2y_1' - y_1y_2'}{(y_2)^2}$$

which is zero exactly when $y_2y_1' - y_1y_2' = 0$, and this is exactly the Wronskian determinant we just saw! Neato.

Examples

Compute the Wronskian of $e^{r_1t},\ e^{r_2t}$ for $r_1 \neq r_2$

Compute the Wronskian of $e^{rt},\ te^{rt}$

Compute the Wronskian of $\cos^2(\theta),\ 1 + \cos 2\theta$