MAT 22B Lecture Notes

Wednesday, 5 August 2015

Terminology

A *differential equation* (DE) is an equation involving a function and its derivative(s). A *solution* to a differential equation is a function that makes the differential equation true. This point is worth over-stressing: to *solve* a differential equation means to find a *function*. For instance, the function $y = e^t$ is a solution of the differential equation $y' = y$, and the function $y = t^2$ is a solution to the differential equation $y' = 2t$.

Clearly, differential equations may have many solutions. For instance, Ae^t is a solution to $y' = y$ for any real number A, and $y = t^2 + c$ is a solution to $y' = 2t$ for any real number c. An expression that captures all possible solutions to a DE at once is called the *general solution* to that DE.

A DE expresses one piece of information we can know about a function. We might also know the value of the unknown function at some point, like $y(0) = 1$. Such a piece of information is called an *initial condition*. A DE together with an initial condition is called an *initial value problem* (IVP). As you can probably guess, the solution to an IVP is a solution to the DE which also satisfies the initial condition.

Classification of DE's

A differential equation is an equation involving a function and its derivative(s). That is extremely general, and so there's no general purpose way to solve every DE. However, there are techniques that work for large classes of DE's, so it's important to learn how to identify them. The following introduces some of the fundamentals of classifying DE's

Partial vs. Ordinary

A partial differential equation (PDE) is a DE which contains partial derivatives. This can only happen if the unknown function is a function of more than one variable (for instance, x, y, and z positions in space, or space and time). An ordinary differential equation (ODE) is a DE where the unknown function is a function of only one variable, and so can only contain "ordinary" derivatives. There is a lot of theory for solving ODE's that just plain doesn't work for PDE's. The theory of PDE's is beautiful and deep and massively applicable to the real world, but we will only be thinking about ODE's in this class.

Order

The order of a differential equation is the order of the highest order derivative in the equation. For instance, $y' + y = 0$ is first order, $y'' + y = 0$ is second order, $(y')^2 = t$ is first order, and so on. First order DE's are nice because, among other things, they can often be rearranged and solved directly by integration. It is also easier to design computer algorithms to numerically solve first order DE's.
Typically, the general solution to an n^{th} order ODE has n free parameters. One way to think about why this is the case is to consider solving $y^{(n)} = g(t)$. To arrive at $y(t)$, we need to integrate n times, and each time, we need to include another constant of integration.

Linear vs. Nonlinear

The notion of linearity of a DE is a strange one until you get used to it, but it is important to understand what it means and how to identify it. The most general form of an n^{th}-order linear ODE is

$$a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)} = g(t)$$

where $y^{(n)}$ denotes the n^{th} derivative of y. Note that the *coefficients* of the y terms, the $a_k(t)$, are, in general, *functions of* t. If the coefficients do not actually depend on t, then we say that the equation has constant coefficients. This means that something like $y' + t^2 y = 0$ is still a linear ODE. The key point is that the unknown function y appears only as itself or one of its derivatives - not, for instance, as y^2, or $\sin(y)$, or yy', and so on.
The key fact that linearity gives us is the following; if we abbreviate the left-hand side above as

$$L[y] = a_0(t)y + a_1(t)y' + a_2(t)y'' + \cdots + a_n(t)y^{(n)}$$

then the function L has the properties that $L[y_1 + y_2] = L[y_1] + L[y_2]$, and $L[ky] = kL[y]$, for any (smooth) functions y, y_1, y_2 and constant k. This means, in a certain sense, that solving $L[y] = g$ has a lot in common with solving equations like $A\vec{x} = \vec{b}$ in linear algebra. More on this later.
A final remark about this setup; if the function $g(t)$ on the right-hand side is zero, then we say that the equation is homogeneous, and otherwise, we say that it is non-homogeneous. Again, the difference is the same as the difference between solving $A\vec{x} = \vec{0}$ and solving $A\vec{x} = \vec{b}$

Examples

$$x'' + bx' + kx = 0$$

Second order (x''), linear, homogeneous

$$yy' = 2t$$

First order (y'), nonlinear (yy'), non-homogeneous ($2t$)

$$\frac{d^4x}{dt^4} + \sin(t)\frac{d^2x}{dt^2} - 4x = \cos(t)$$

Fourth order, linear, non-homogeneous ($\cos(t)$)

Separable Equations

Separable equations are the nicest of all ODE's. They are called separable because it is possible to "separate" the independent and dependent variables - that is, you can gather all the t's together and all the y's together. These are equations of the form

$$f(y)\frac{dy}{dt} = g(t)$$

The thing that almost everyone does, but is not fully rigorous by itself, is to "multiply through by dt and integrate", so you get

$$\int f(y)dy = \int g(t)dt$$

and after integrating, you have some function of y equal to some other function of t. Sometimes, you can then solve for y as a function of t; when this is possible, we say that we have found an *explicit* solution. Otherwise, we leave it as is, and say we've found an *implicit* solution. For instance, we might end up with something like $y^2 = 1 - t^2$, which defines a circle in the (t, y)-plane, and we can't write down a single-valued function $y(t)$ that traces out the same points.
The way to make the "multiply by dt" step rigorous is the following. Say $F(y)$ is an antiderivative of $f(y)$, so that $\frac{dF}{dy} = f$. Since y is then a function of t, we can differentiate F with respect to t using the chain rule:

$$\frac{dF}{dt} = \frac{dF}{dy}\frac{dy}{dt} = f(y)\frac{dy}{dt}$$

which is the left-hand side of the equation we began with. So we have

$$\frac{dF}{dt} = g(t)$$

and now we can integrate each side with respect to t to get

$$F(y) = \int g(t)dt$$

but $F(y) = \int f(y)dy$. So we end up with the same formula we got by pushing around the not-quite-well-defined infinitesimal dt. Indeed, the fact that this works is the reason that notation was invented.

Examples

$$\frac{dy}{dt} = -y^2 t \implies y = \frac{1}{t^2 + c}$$

$$\frac{dy}{dt} = yt^2 \implies y = Ae^{t^3/3}$$

$$\frac{dy}{dt} = (y-1)t \implies y = Ae^{t^2/2} + 1$$

First Order Linear Equations - the Integrating Factor

The next class of ODE's we'll consider is first order linear equations. According to the general form we saw before, these can all be written as

$$a_0(t)y + a_1(t)y' = g(t)$$

However, we'll rewrite it slightly by dividing through by $a_1(t)$, to get

$$y' + p(t)y = g(t)$$

where $p = \frac{a_0}{a_1}$. You may be concerned about if $a_1(t) = 0$ for some t, and you should be - this situation presents its own difficulties that we aren't ready to deal with yet. So for now, let's assume that $a_1(t) \neq 0$ for all t.

The key insight to the integrating factor method is to make the left-hand side look like the derivative of a single thing, so that we can simply integrate and solve for y. It turns out that we can do that! To see how, suppose we multiply both sides of the equation by some function $\mu(t)$:

$$\mu(t)y' + \mu(t)p(t)y = \mu(t)g(t)$$

Now, wouldn't it be nice if the left-hand side were the derivative of, say, $\mu(t)y$? Let's see if that is possible:

$$\frac{d}{dt}\left[\mu(t)y\right] = \mu(t)y' + \mu'(t)y$$

The above expression is equal to the left-hand side of our DE as long as $\mu'(t) = \mu(t)p(t)$. Assuming that we've found such a $\mu(t)$, then we can rewrite our DE as

$$(\mu y)' = \mu g$$

So,

$$\mu(t)y(t) = \int \mu(t)g(t)dt \implies y(t) = \frac{\int \mu(t)g(t)dt}{\mu(t)}$$

keeping in mind that the antiderivative above can include any arbitrary constant of integration. So we've found our solution! The only missing piece is the function μ, which we can determine from the equation $\mu' = p\mu$. Noting that this is a separable equation, we get

$$\frac{d\mu}{dt} = \mu(t)p(t) \implies \int \frac{d\mu}{\mu} = \int p(t)dt \implies \mu(t) = A\exp\left(\int p(t)dt\right)$$

for any constant A. We will generally choose $A = 1$ because any μ with $\mu' = p\mu$ will do the trick.

Examples

$$y' + \frac{2}{t}y = \frac{\cos(t)}{t^2} \quad y(\pi) = 0,\ t > 0 \implies y = \frac{\sin(t)}{t^2}$$

$$y' + 3t^2 y = t^2, \quad y(0) = 1 \implies y = \frac{1}{3} + \frac{2}{3}e^{-t^3}$$

$$ty' + (t+1)y = t, \quad y(\ln 2) = 1,\ t > 0 \implies y = 1 - \frac{1}{t} + \frac{2}{te^t}$$