MAT 22B - Lecture Notes

4 September 2015

Solving Systems of ODE

Last time we talked a bit about how systems of ODE arise and why they are nice for visualization. Now we'll talk about the basics of how to solve linear systems, with some linear algebra review along the way.

The model system we will consider is first order, linear, homogeneous, and has constant coefficients. That is, we can write the system as

$$\frac{d\vec{y}}{dt} = \mathbf{A}\vec{y}$$

where $\vec{y}$ is an n-dimensional vector depending on time t and $\mathbf{A}$ is a fixed $n \times n$ matrix.

Eigenvectors and Eigenvalues

As we noticed before, eigenvector/eigenvalue pairs provide us with solutions to the ODE above. To recall, say $\vec{y}_0$ is an eigenvector of $\mathbf{A}$ with eigenvalue λ. That is, we have

$$\mathbf{A}\vec{y}_0 = \lambda\vec{y}_0$$

so that $\mathbf{A}$ acts on $\vec{y}_0$ by simply "stretching" it by a factor λ. We might sometimes refer to $(\vec{y}_0, \lambda)$ as an *eigenpair*, to emphasize that they come together - the eigenvalue means nothing unless we know the corresponding eigenvector, and vice versa.

Given such an eigenpair, we can construct a vector-valued function of t, by the formula

$$\vec{y}(t) = e^{\lambda t}\vec{y}_0$$

If you visualize this function as a point moving around n-dimensional space through time, then it traces out a straight line, which is the ray spanned by $\vec{y}_0$. If $\lambda > 0$, then $\vec{y}(t)$ will get longer (i.e. move away from the origin) as t increases, and if $\lambda < 0$, then $\vec{y}(t)$ will move towards the origin as t increases.

The key thing about this function is that it is a solution to the ODE. To see this, we can compute each side of the equation. First, the derivative is

$$\frac{d\vec{y}}{dt} = \frac{d}{dt}(e^{\lambda t})\vec{y}_0 = \lambda e^{\lambda t}\vec{y}_0$$

Next, the matrix $\mathbf{A}$ acts on this function as

$$\mathbf{A}\vec{y}(t) = \mathbf{A}e^{\lambda t}\vec{y}_0 = e^{\lambda t}\mathbf{A}\vec{y}_0 = \lambda e^{\lambda t}\vec{y}_0$$

where we have used that matrix multiplication is linear (i.e. that $\mathbf{A}(c_1\vec{x}_1 + c_2\vec{x}_2) = c_1\mathbf{A}\vec{x}_1 + c_2\mathbf{A}\vec{x}_2$). So, viola! The function $\vec{y}(t)$ satisfies the ODE.

Fundamnetal Sets of Solutions - Just like before

Based on the line of reasoning we followed solving first and second order equations, the next logical step should be to determine if we've captured *all* the solutions to the ODE. This question warrants thinking about the set of solutions we can expect to find.

First, when our system is linear and homogeneous, the set of solutions form a vector space - that is, we can take linear combinations of solutions and get another solution. In symbols, we have that for any constants c_1, c_2 and any two solutions $\vec{y}_1, \vec{y}_2$ to the ODE, the function $\vec{y} = c_1\vec{y}_1 + c_2\vec{y}_2$ is also a solution. The proof works just how it did in the scalar case:

$$\begin{aligned}(c_1\vec{y}_1 + c_2\vec{y}_2)' &= c_1\vec{y}_1' + c_2\vec{y}_2' \\ &= c_1\mathbf{A}\vec{y}_1 + c_2\mathbf{A}\vec{y}_2 \\ &= \mathbf{A}\left(c_1\vec{y}_1 + c_2\vec{y}_2\right)\end{aligned}$$

Since the set of solutions forms a vector space, it has a basis. If we find such a basis, say $(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n)$, then *every* solution will be of the form

$$c_1\vec{y}_1 + c_2\vec{y}_2 + \ldots + c_n\vec{y}_n$$

for some choice of contants (c_i). The key issue now is to figure out where to look for such a basis, and how to know when we've found it.

The discussion of eigenvectors above indicates that this might be a fruitful place to look for solutions that might form a basis. To check whether we've gotten everything, we'll do something just like we did in the second-order scalar case: make sure that we can meet any initial condition we might want to impose. That is, given some collection of solutions $(\vec{y}_1, \vec{y}_2, \ldots, \vec{y}_n)$ and an initial condition $\vec{y}(0) = \vec{y}_0$, we want to make sure there exist constants $(c_1, \ldots, c_n)$ so that

$$c_1\vec{y}_1(0) + \ldots + c_n\vec{y}_n(0) = \vec{y}_0$$

From linear algebra, this is the case precisely if the set of vectors $\{\vec{y}_1(0), \vec{y}_2(0), \ldots, \vec{y}_n(0)\}$ is linearly independent. Equivalently, we can write the equation above in matrix form as

$$\begin{pmatrix} y_1^1(0) & y_2^1(0) & \ldots & y_n^1(0) \\ y_1^2(0) & y_2^2(0) & \ldots & y_n^2(0) \\ \vdots & \vdots & & \vdots \\ y_1^n(0) & y_2^n(0) & \ldots & y_n^n(0) \end{pmatrix} \begin{pmatrix} c_1 \\ c_2 \\ \vdots \\ c_4 \end{pmatrix} = \begin{pmatrix} y_0^1 \\ y_0^2 \\ \vdots \\ y_0^n \end{pmatrix} \qquad \text{or} \qquad \mathbf{\Psi}(0)\vec{c} = \vec{y}_0$$

where superscripts specify components of the various vectors, i.e.

$$\vec{y}_1(0) = \begin{pmatrix} y_1^1(0) \\ y_1^2(0) \\ \vdots \\ y_1^n(0) \end{pmatrix}$$

So, to check if the matrix equation above as a solution $\vec{c}$ for every initial condition vector $\vec{y}_0$, we can check if the determinant of the matrix $\mathbf{\Psi}(0)$ is or is not zero.

The Phase Plane

Because we're rather short on time to give the general treatement of systems of ODE, we're skipping ahead to chapter 9 to see how the "linear, constant coefficient, homogeneous" claptrap we've been peddling all session is really more powerful and informative than you might initially think.

For the remainder of the discussion, we'll focus on a two-dimensional ODE. That is, solutions will be functions of time which take values in the plane. To start, we'll consider systems of the form

$$\vec{x}' = \mathbf{A}\vec{x}$$

and consider the different *kinds* of solutions we can get, based on the eigenvalues and eigenvectors of $\mathbf{A}$. To be clear, $\vec{x}$ is a two-dimensional vector, and $\mathbf{A}$ is a 2×2 matrix.

Notice that regardless of $\mathbf{A}$, the function $\vec{x}(t) = \vec{0}$ is always a solution of the system. A boring solution, but a solution. Recalling our discussion of autonomous equations, we say that $\vec{x} = \vec{0}$ is an *equilibrium point* of the system. In our discussion before, we talked about the stability of equilibria, in terms of whether nearby trajectories move towards or away from the equilibrium. We'll do the same thing in the two-dimensional case, and find that there are decidedly more possibilities for what kind of stability an equilibrium can possess (and there become yet *more* in higher dimensions).

Real, Distinct Eigenvalues

Let's say $\mathbf{A}$ has two distinct, real eigenvalues λ_1 and λ_2 with eigenvectors $\vec{\xi_1}$ and $\vec{\xi_2}$, respectively. Then the function

$$\vec{x} = c_1\vec{\xi}_1 e^{\lambda_1 t} + c_2\vec{\xi}_2 e^{\lambda_2 t}$$

is a solution to the system for any c_1, c_2. How can we visualize this? Well, that depends. If λ_1 and λ_2 are both positive, then every solution goes away to infinity as $t \to \infty$. In this case, we say that the equilibrium $\vec{x} = \vec{0}$ is an *unstable node*. Likewise, if both λ_1 and $\lambda_2$are negative, then every solution will approach $\vec{0}$ as $t \to \infty$. In this case, we say that the equilibrium $\vec{x} = \vec{0}$ is a *stable node*.

If λ_1 and λ_2 have opposite signs, however, the behavior is more interesting. Say $\lambda_1 > 0$ and $\lambda_2 < 0$. Then if $c_1 \neq 0$ in the above expression, then $\vec{x}(t)$ contains a term that will grow very large as $t \to \infty$ (namely, $e^{\lambda_1 t}$). However, if $c_1 = 0$, then the only term left is $c_2\vec{\xi}_2 e^{\lambda_2 t}$, which approaches zero as $t \to \infty$.

This is a situation we don't see for one-dimensional first order equations. For many initial conditions, the trajectory will run off to infinity, but for some initial conditions, the trajectory will go towards the fixed point $\vec{x} = \vec{0}$. In this case, we say that the equilibrium is a *saddle* or *saddle point*. To see why, consider rolling a ball on a saddle-shaped surface. The ball will settle down into the middle if rolled *exactly* at the right angle, but will fall off and move away from the middle otherwise.

Repeated Eigenvalues

Degenerate node. The interesting/complicated ratio is too low to talk about right now.

Complex Eigenvalues

We can do some abstract junk with complex solutions and yadda yadda but there's a simpler way to deal with the system in our current (2D) setting. Consider the system

$$\vec{x}' = \begin{pmatrix} \lambda & \mu \\ -\mu & \lambda \end{pmatrix} \vec{x}$$

You can check that the matrix above has eigenvalues $\lambda \pm i\mu$. In scalar form, we have

$$\begin{aligned} x_1' &= \lambda x_1 + \mu x_2 \\ x_2' &= -\mu x_1 + \lambda x_2 \end{aligned}$$

Since your brain should by now be primed to think rotation whenever you see complex numbers, let's convert to polar coordinates. Recall

$$r^2 = x_1^2 + x_2^2, \quad \tan\theta = \frac{x_2}{x_1}$$

We're going to convert the system into ODEs for r and θ. Differentiating the first equation with respect to t we get

$$\begin{aligned} 2rr' &= 2x_1x_1' + 2x_2x_2' \\ rr' &= x_1\left(\lambda x_1 + \mu x_2\right) + x_2\left(-\mu x_1 + \lambda x_2\right) \\ rr' &= \lambda\left(x_1^2 + x_2^2\right) = \lambda r^2 \end{aligned}$$

so $r' = \lambda r$. What a relief! Now for θ

$$\begin{aligned} \left(\sec^2\theta\right)\theta' &= \frac{x_1x_2' - x_2x_1'}{x_1^2} \\ \left(\sec^2\theta\right)\theta' &= \frac{1}{x_1^2}\left(x_1\left(-\mu x_1 + \lambda x_2\right) - x_2\left(\lambda x_1 + \mu x_2\right)\right) \\ \left(\sec^2\theta\right)\theta' &= \frac{-\mu(x_1^2 + x_2^2)}{x_1^2} = -\mu\frac{r^2}{x_1^2} \end{aligned}$$

By geometry, $\cos\theta = \frac{x_1}{r}$, so this reduces to $\theta' = -\mu$. WAHOO.

As if by magic, these coupled differential equations for x_1 and x_2 have turned into a pair of *uncoupled* equations for r and θ. We can solve them in a moment;

$$r(t) = r_0 e^{\lambda t}, \quad \theta(t) = \theta_0 - \mu t$$

In other words, the point $\vec{x}(t)$ rotates around the origin at frequency μ, and moves either towards or away from the origin in a way dictated by λ. If $\lambda > 0$, then all trajectories go off to infinity as $t \to \infty$, and we say that the origin is an *unstable spiral*. Conversely, if $\lambda < 0$, all trajectories fall into the origin, and we call the origin a *stable spiral*.

Pure Imaginary Eigenvalues

You may notice that we left out the case $\lambda = 0$ above. It's not that different from the spiral case, except that trajectories neither move towards nor away from the origin. In this case the origin is called a *center*. It has "neutral" stability. This happens with an undamped mass on a spring - oscillations continue forever and ever, without dying out or blowing up to infinity.

Approximating a Nonlinear ODE by a Linear ODE near an equilibrium point

If I have a nonlinear two-dimensional ODE

$$\vec{x}' = \vec{F}(\vec{x})$$

then I can find its equilibria, that is, vectors $\vec{x}^*$ such that $\vec{F}(\vec{x}^*) = 0$, so that a trajectory that starts there will never leave. It turns out that if $\vec{F}$ is smooth enough, we can classify its equilibria into exactly the same categories as we just found above. This is because of the multi-dimensional Taylor's formula.

The idea is to approximate $\vec{F}$ by a simpler function in the vicinity of some interesting behavior - in this case, an equilibrium. Taylor's formula does exactly that. We have

$$\vec{F}(\vec{x}) = \vec{F}(\vec{x}^*) + \frac{d\vec{F}}{d\vec{x}}(\vec{x} - \vec{x}^*) + \mathcal{O}\left((\vec{x} - \vec{x}^*)^2\right)$$

where the first derivative is the *Jacobian*, defined as

$$\frac{d\vec{F}}{d\vec{x}} = \begin{pmatrix} \frac{dF_1}{dx_1} & \frac{dF_1}{dx_2} \\ \frac{dF_2}{dx_1} & \frac{dF_2}{dx_2} \end{pmatrix}$$

and all the functions are to be evaluated at the point $\vec{x}^*$ (where we are centering the Taylor expansion). Given also that $\vec{F}(\vec{x}^*) = \vec{0}$, we have that for $\vec{x}$ near $\vec{x}^*$,

$$\vec{x}' \approx \mathbf{J}\vec{x}$$

where $\mathbf{J}$ is the Jacobian matrix we just defined. So near the equilibrium, this ODE looks just like a homogeneous, linear system with constant coefficients! All the analysis above then carries through and we can talk about equilibria being saddles, nodes, spirals, and so on.

If this revs your engine, take MAT 119 next time it's offered. The A quarter starts with this stuff, and the B quarter gets into chaos theory.