22B Synopsis

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First Order Equations

Linear

A linear first-order differential equation has the form

$$y' + p(t)y = g(t)$$

for some functions $p(t)$ and $g(t)$. The general solution is obtained using an integrating factor:

$$y = \frac{1}{\mu(t)}\left[\int_{t_0}^{t} \mu(s)g(s)ds + c\right], \quad \text{where } \mu(t) = \exp\left(\int p(t)dt\right)$$

Separable

Some nonlinear first-order equations are of a form that allows them to be solved exactly by integration. These are equations that can be written in the form

$$f(y)\frac{dy}{dt} = g(t)$$

Note that f need not depend linearly on y. If a differential equation can be written in the form above, we can solve it by integration

$$f(y)dy = g(t)dt \implies \int f(y)dy = \int g(t)dt$$

This will give an algebraic equation involving both t and y; this equation will specify a curve in the ty-plane, but it will not, in general, be possibe to write y as a function of t explicitly.

Autonomous

A differential equation is called *autonomous* if it can be written in the form

$$y' = f(y)$$

Every autonomous DE is separable, and can be solved by the following integral

$$\int \frac{dy}{f(y)} = \int dt$$

Solutions to first-order autonomous equations cannot oscillate. This fact can be seen by looking at the phase line.

Second Order

Classification and Structure of Solution Space

A *linear* second order DE is of the form

$$L[y] = y'' + p(t)y' + q(t)y = g(t)$$

where L denotes the differential operator that takes in the (twice differentiable) function y and outputs the function $y'' + p(t)y' + q(t)y$. The defining property of a linear DE is that

$$L[\alpha y_1 + \beta y_2] = \alpha L[y_1] + \beta L[y_2]$$

for any constants α, β, and any (twice differentiable) functions y_1, y_2. A linear, *homogeneous* second order DE is of the form

$$L[y] = y'' + p(t)y' + q(t)y = 0$$

The most important property of a homogeneous linear DE is that for any two functions y_1 and y_2 that are solutions, any linear combination of y_1 and y_2 is also a solution, because

$$L[\alpha y_1 + \beta y_2] = \alpha L[y_1] + \beta L[y_2] = 0$$

A pair of functions $\{y_1, y_2\}$ that are both solutions to a linear, homogeneous DE is said to comprise a *fundamental set of solutions* if *any* solution y the DE can be written in the form $c_1y_1 + c_2y_2$, for some constants c_1, c_2. To check if a given pair of functionss comprises a fundamental set of solutions, we compute their *Wronskian*, which is defined as

$$W[y_1, y_2] = \begin{vmatrix} y_1 & y_2 \\ y_1' & y_2' \end{vmatrix} = y_1y_2' - y_2y_1'$$

The Wronskian of two solutions is nonzero if and only if they comprise a fundamental set of solutions. This is a consequence of considering the problem of determining c_1 and c_2 such that $c_1y_1 + c_2y_2$ satisfies an arbitrary pair of initial conditions, $y(t_0) = a, y'(t_0) = b$.

A DE is said to have *constant coefficients* if the coefficient functions $p(t)$ and $q(t)$ are constants. The most general form of a linear, homogeneous, constant coefficient, second order DE is

$$ay'' + by' + cy = 0$$

Its solutions depend on the roots of the characteristic polynomial

$$ar^2 + br + c = 0$$

The roots may be either real and distinct, complex, or repeated. In these cases the fundamental sets of solutions can be taken to be:

- Real and distinct: $y_1 = e^{r_1t}$, $y_2 = e^{r_2t}$
- Repeated: $y_1 = e^{rt}$, $y_2 = te^{rt}$
- Complex: $r = a \pm bi$, then $y_1 = e^{at}\cos bt$, $y_2 = e^{at}\sin bt$

Nonhomogeneous Equations

If $y_p(t)$ is *any* solution to the nonhomogeneous equation

$$y'' + p(t)y' + q(t)y = g(t)$$

then *any other* solution is of the form $y(t) = y_p(t) + y_c(t)$, where $y_c(t)$ is a solution to the *corresponding homogeneous equation*, which is obtained by replacing $g(t)$ by zero;

$$y'' + p(t)y' + q(t)y = 0$$

We say that y_p is a *particular* solution, and y_c is a *complementary* solution. If we know a fundamental set of solutions to the corresponding homogeneous equation, we simply need to find *one* solution to the nonhomogeneous equation in order to know the general solution. We have studied two main methods for doing this:

Undetermined Coefficients

This is a guess-and-check method that is used when the corresponding homogeneous equation has constant coefficients, and the nonhomogeneous term, $g(t)$, is "simple" - this means sums or products of polynomials, exponentials, or trig functions. Based on the form of $g(t)$, we make a guess at $y_p(t)$ that includes some number of undetermined coefficients, substitute this guess into the DE, and solve for the values of these coefficients that make the equation true. The following table outlines what to guess given various $g(t)$:

$g(t)$	Guess at $y_p(t)$
$P_n(t) = a_n t^n + \cdots + a_1 t + a_0$	$t^s \left(A_n t^n + \cdots + A_1 t + A_0\right)$
$P_n(t)e^{at}$	$t^s \left(A_n t^n + \cdots + A_1 t + A_0\right) e^{at}$
$P_n(t)e^{at} \begin{cases} \sin bt \\ \cos bt \end{cases}$	$t^s e^{at} \left[(A_n t^n + \cdots + A_1 t + A_0)\cos bt + (B_n t^n + \cdots + B_1 t + B_0)\sin bt\right]$

where s denotes the smallest integer (either 0,1, or 2) that ensures that no term in the guess is a solution to the corresponding homogeneous equation.

Variation of Parameters

This is an explicit method that is applicable in more general situations than undetermined coefficients. However, it requires that we first know a fundamental set of solutions $\{y_1, y_2\}$ to the corresponding homogeneous equaiton. In this case, the general solution to

$$y'' + p(t)y' + q(t)y = g(t)$$

is

$$y = -y_1(t) \int \frac{y_2(s)g(s)}{W[y_1, y_2](s)} ds + y_2(t) \int \frac{y_1(s)g(s)}{W[y_1, y_2](s)} ds$$

where the indefinite integrals denote any antiderivative. The freedom in choice of antiderivative (i.e. choice of constant of integration) is the same as the freedom we have to add any complementary solution.

Laplace Transform

The Laplace transform is a gadget that eats functions and spits out other functions. It is defined by the formula

$$\mathcal{L}[f(t)](s) = \int_0^\infty e^{-st} f(t)dt$$

whenever the integral converges. Typically when a function is denoted by a lowercase letter, its Laplace transform is denoted by the corresponding uppercase letter. For example, we typically write $\mathcal{L}[y(t)] = Y(s)$. The Laplace transform has the important property that if f is differentiable and both f and f' have Laplace transforms, then

$$\mathcal{L}[f'] = s\mathcal{L}[f] - f(0)$$

which is a consequence of integration by parts. Applying this formula n times gives

$$\mathcal{L}[f^{(n)}] = s^n\mathcal{L}[f] - s^{n-1}f(0) - s^{n-2}f'(0) - \cdots - sf^{(n-2)}(0) - f^{(n-1)}(0)$$

where $f^{(n)}$ denotes the n^{th} derivative of f. This property allows us to transform a linear, constant coefficient initial value problem (of any order!) into an algebraic equation which we can solve for $Y(s)$, the Laplace transform of the solution. We find the solution itself by inverting the transform by use of a table. This step will often involve some straightforward, but tedious, algebra; for instance, partial fraction decomposition and completing the square.

Systems of DE

Just like we can have systems of algebraic equations, whose solutions are lists of numbers, we can have systems of differential equations, whose solutions are lists of functions. We will often use vector notation for such lists of functions;

$$\mathbf{x}(t) = \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix}$$

A generic system of DE then has the form

$$\mathbf{x}'(t) = \mathbf{F}(\mathbf{x}, t)$$

where $\mathbf{F}$ is a vector-valued function, depending possibly on both $\mathbf{x}$ and t. Just as in the scalar case, we can classify systems of DE.

A system of DE is *linear* if it can be written as

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t) + \mathbf{g}(t)$$

where $A(t)$ is a matrix, and $\mathbf{g}(t)$ is a vector (both of which may depend on t). A system of DE is called *homogeneous* if the $\mathbf{g}(t)$ term is zero; that is,

$$\mathbf{x}'(t) = A(t)\mathbf{x}(t)$$

Linearity and homogeneity for systems have the same consequences as they do for scalar DE's. In particular, any linear combination of solutions is also a solution. Likewise, we can talk about a *fundamental set of solutions* to a system of DE. This is a set $\{\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)\}$ of n different solutions to the system so that *any* solution $\mathbf{x}(t)$ can be written as

$$\mathbf{x}(t) = c_1\mathbf{x}^{(1)}(t) + \cdots + c_n\mathbf{x}^{(n)}(t)$$

for some set of constants $c_1 \ldots c_n$. We can check if a list of solutions has this property by computing their Wronskian, defined as the determinant of the $n \times n$ matrix, each of whose columns is one of the solutions;

$$W[\mathbf{x}^{(1)}(t), \mathbf{x}^{(2)}(t), \ldots, \mathbf{x}^{(n)}(t)] = \begin{vmatrix} \mathbf{x}^{(1)}(t) & \mathbf{x}^{(2)}(t) & \ldots & \mathbf{x}^{(n)}(t) \\ \downarrow & \downarrow & & \downarrow \end{vmatrix}$$

This is a consequence of considering the problem of finding constants $c_1, \ldots, c_n$ so that $\mathbf{x}(t) = \sum c_i\mathbf{x}^{(i)}(t)$ satisfies an arbitrary set of initial conditions $\mathbf{x}(t_0) = \mathbf{a}$. Finally, a system of DE is said to have *constant coefficients* if the coefficient matrix A does not depend on time. Linear, homogeneous systems with constant coefficients are those that we will figure out how to solve.

Finding solutions to Systems of DE

As stated above, we'll consider solving first order, linear, homogeneous systems with constant coefficients. We begin, as we have done many times before, by guessing the form of the solution. In this case the guess is

$$\mathbf{x}(t) = \mathbf{a}e^{\lambda t}$$

where $\mathbf{a}$ is some constant vector and λ is a constant scalar. If such a function is a solution of the system $\mathbf{x}'(t) = A\mathbf{x}(t)$, it follows that $\mathbf{a}$ is an *eigenvector* of A with *eigenvalue* λ. Hence the spectrum of A (that is, its eigenvalues and eigenvectors) determine the behavior of solutions. We can break down the possibilities into cases:

- All λ are real and distinct.
  - In this case, the corresponding eigenvectors are linearly independent, and the functions $\mathbf{x}^{(i)} = \mathbf{a}^{(i)}e^{\lambda_i t}$ form a fundamental set of solutions. Here $\mathbf{a}^{(i)}$ denotes an eigenvector of A having eigenvalue λ_i.
- Two or more λ are complex
  - When the matrix A has real entries, complex eigenvalues occur in conjugate pairs, and the corresponding eigenvectors are also complex, and occur in conjugate pairs. We can then write a complex-valued solution

    $$\mathbf{x}(t) = (\mathbf{a} + i\mathbf{b})e^{(\lambda + i\mu)t} = \mathbf{u}(t) + i\mathbf{v}(t)$$

    and invoke the principle of superposition to conclude that each of $\mathbf{u}(t)$ and $\mathbf{v}(t)$ is a real-valued solution.
- Some λ are repeated
  - This is analogous to the case of repeated roots of the characteristic equation, and involves some tricks we haven't had time to cover.

The neatest thing we just barely didn't get to

Recall that in the first week of class, we saw that the equation

$$y' = ay$$

has the solution

$$y = y_0 e^{at}$$

where $y_0 = y(0)$ is an initial condition. Analogously, for a diagonalizable matrix A, the system

$$\mathbf{x}'(t) = A\mathbf{x}(t)$$

has the solution

$$\mathbf{x}(t) = e^{At}\mathbf{x}^0$$

where $\mathbf{x}^0$ is the vector of initial conditions; $\mathbf{x}(0) = \mathbf{x}^0$. At this point you should be seriously wigged out - that is a *matrix* inside an exponent. But it's fine; we can define it using the Taylor series for e^x. Recall:

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!}$$

This series has an infinite radius of convergence. Since we know how to raise a matrix to a power (just multiply it by itself), and mutliply it by scalars (e.g. $^1/_{n!}$), we can insert the matrix At in place of x, to get

$$e^{At} = \sum_{n=0}^{\infty} \frac{(At)^n}{n!} = \sum_{n=0}^{\infty} A^n \frac{t^n}{n!}$$

Now we can make sense of the function $\mathbf{x}(t) = e^{At}\mathbf{x}^0$. Differentiating it with respect to t gives

$$\frac{d}{dt}\left[e^{At}\mathbf{x}^0\right] = \frac{d}{dt}\left[\sum_{n=0}^{\infty} A^n \frac{t^n}{n!}\right]\mathbf{x}^0 = \left(\sum_{n=0}^{\infty} A^n \frac{d}{dt}\left[\frac{t^n}{n!}\right]\right)\mathbf{x}^0$$

The $n = 0$ term in the sum vanishes, and we're left with

$$\frac{d}{dt}\mathbf{x}(t) = \sum_{n=1}^{\infty} A^n \frac{t^{n-1}}{(n-1)!}\mathbf{x}^0 = \sum_{n=0}^{\infty} A^{n+1} \frac{t^n}{n!}\mathbf{x}^0$$

where the second equality comes from shifting the index. Factoring out a factor of A, we get

$$\frac{d}{dt}\mathbf{x}(t) = A\sum_{n=0}^{\infty} A^n \frac{t^n}{n!}\mathbf{x}^0 = Ae^{At}\mathbf{x}^0 = A\mathbf{x}(t)$$

So the claimed solution indeed satisfies the system of DE! It is also easy to check that $\mathbf{x}(0) = \mathbf{x}^0$ as advertised.