Entrainment of Coupled Phase Oscillators

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UC DAVIS
MATHEMATICS

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Los Alamos

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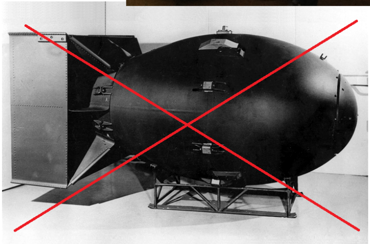

Outline

1. Motivation and Background
   - Collective behavior meets Control
   - Example System
   - Entrainment
   - Coupled Oscillators
2. Present Work
   - Forced, Coupled Oscillators
   - Linear Stability Analysis
   - Conclusions
3. Future Work
   - Structured Coupling
   - Multiple Timescales

Complexity

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- Global behavior not obvious from local dynamics
- "Magnets"

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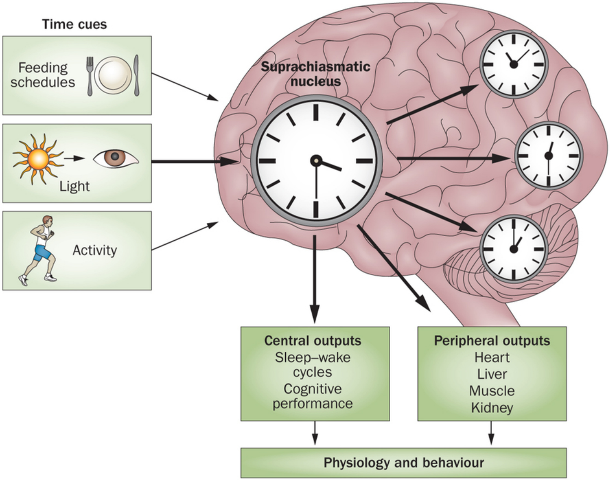
- As few measurements as possible $\Longrightarrow$ *open-loop control*

$\Longrightarrow$ **Goal**: Study a simple system where we can analyze the interplay of open-loop control and collective behavior.

$\Longrightarrow$ **Q**: Which simple system?

Inspiration : Biological Rhythms



[Videnovic *et al.*, 2014]

Model Class

Desired features:

- Many coupled units with different intrinsic, periodic behavior
- Subject to external forcing with constant period
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Model System : Population of nonlinear oscillators subject to *forcing* and *coupling*

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where $\Delta\omega = \omega - \Omega$ is the *frequency detuning* and Λ_v is the *interaction function*:

$$\Lambda_v(\varphi) = \int_0^{2\pi} Z(\varphi + \theta)v(\theta)d\theta$$

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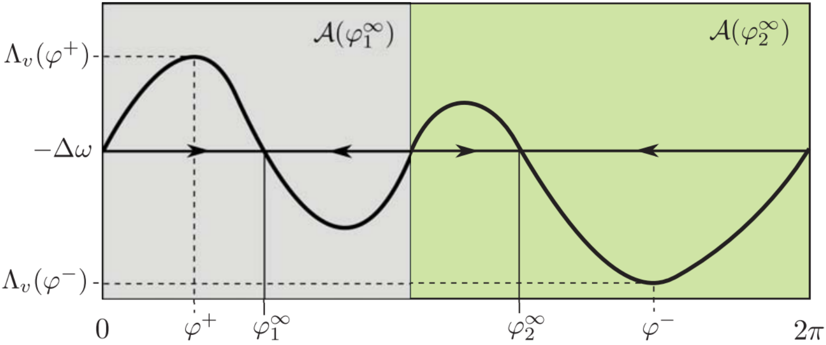
Entrainment

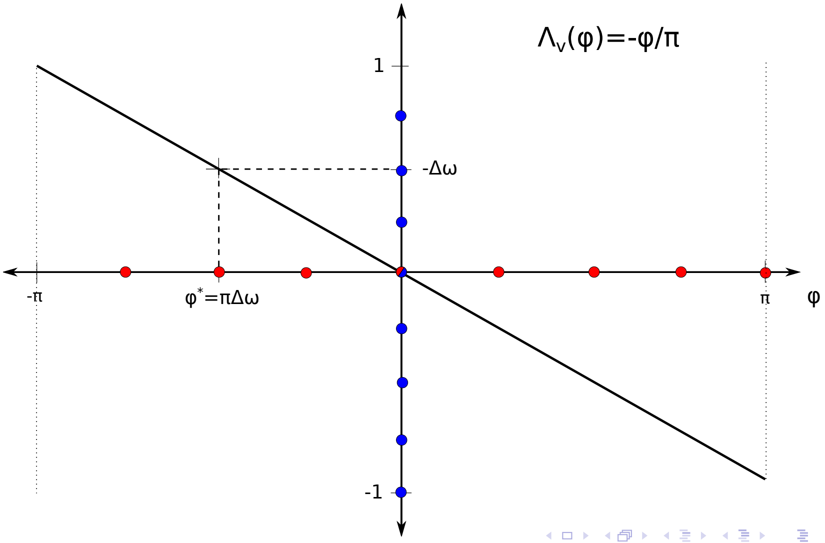
A phase φ^* is a stable fixed point of $\dot{\varphi} = \Delta\omega + \Lambda_v(\varphi)$ if

$$\Delta\omega + \Lambda_v(\varphi^*) = 0 \text{ and } \frac{d\Lambda_v}{d\varphi}(\varphi^*) < 0$$



Zlotnik *et al.* [2016]

Example : Entrained Decoherence



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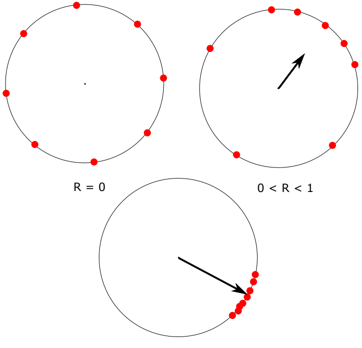
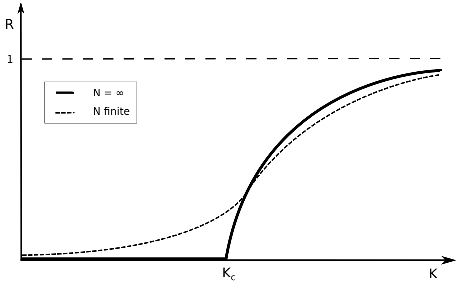
Trade-off between heterogeneity and coupling at *critical coupling strength* $K_c = \frac{2}{\pi g(0)}$

Phase transition in the Kuramoto model





K_c = point when $R = 0$ state becomes *linearly unstable*

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Entrained Decoherence with Coupling

$$\dot{\varphi}_i = \omega_i + \Lambda_v(\varphi_i) + \frac{K}{N}\sum_{j=1}^{N}\sin(\varphi_j - \varphi_i)$$

where $\{\omega_i = \frac{2}{N}i - 1\}$ and $\Lambda_v(\varphi) = -\frac{\varphi}{\pi}$ for $\varphi \in [-\pi, \pi)$.

Decoherence - $\{\varphi_i = \pi\omega_i\}$ - is an $R = 0$ fixed point, by symmetry of the coupling term.

Q: For what values of K is this state stable?

We analyze this model for finite N and in the limit $N \to \infty$

Linear Stability in Finite Dimensions

At the fixed point $\{\varphi_i^* = \pi\omega_i\}$, the Jacobian has matrix elements

$$J_{ij} = \left(\frac{-1}{\pi} - \frac{K}{N} \sum_{k \neq i} \cos(\varphi_k^* - \varphi_i^*) \right) \delta_{ij} + (1 - \delta_{ij}) \frac{K}{N} \cos(\varphi_j^* - \varphi_i^*)$$

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Bound the eigenvalues using the Gershgorin circle theorem:

$$\mathfrak{Re}(\lambda) \leq \frac{-1}{\pi} + K \implies K_c \geq \frac{1}{\pi}$$

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Dynamics are given by a *continuity equation*:

$$\partial_t \rho_\omega + D(v_\omega \rho_\omega) = 0$$

where D is the derivative in the sense of distributions, and $v_\omega = v_\omega(\varphi)$ is the *phase velocity*.

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Linearized dyanmics near $\{\rho_\omega = \delta_{\pi\omega}\}$ can be diagonalized exactly:

$$\sigma(L) = \left\{ \frac{-1}{\pi}, \frac{-1}{\pi} + \frac{K}{2} \right\} \implies K_c = \frac{2}{\pi}$$

Conclusions

In finite dimensions, $K_c \geq \frac{1}{\pi}$; in infinite dimensions, $K_c = \frac{2}{\pi}$

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Despite phase diversity, external forcing has brought the system closer to order.

Structure $\leftrightarrow$ Function

The interplay between connectivity and dynamics has been studied in many contexts

- Resilience to random breakdown: $1 - p_c = \frac{\langle k \rangle}{\langle k^2 \rangle - \langle k \rangle}$
- Analyzing structure via dynamics: PageRank
- Master Stability Function: $\text{MSF} : \sigma(A) \to \{\text{stable, not stable}\}$

In synchronization, studies have focused on:

- Cluster sizes for lattices in the limit $N \to \infty$ [Strogatz and Mirollo, 1988]
- Existence & uniqueness of fixed points [Jadbabaie *et al.*, 2004]
- Paths to synchronization on different topologies [Gómez-Gardenes *et al.*, 2007]
- Correlations, e.g. assortativity [Restrepo and Ott, 2014]

Proposed Work

Questions:

- Given coupling topology, what is the change in stability of decoherence upon driving?
- Is it possible to tune the trade-off between driving and coupling by adjusting coupling topology?

Approaches:

- Numerical simulation
- Linear Stability Analysis
- Mean-field Approximation

Subharmonic Entrainment

It is well known that oscillators can behave coherently without attaining the same frequency. Examples include:

- 4-5 day rodent estrous cycle locking to night-day cycle [Winfree, 2001]
- 2:1 phase locking of body temperature and sleep-wake cycle when deprived of external time cues [Aschoff and Wever, 1981]
- Subharmonic locking of a sensory neuron to periodic inhibitory input [Perkel *et al.*, 1964]

Subharmonic entrainment of a single oscillator is well described mathematically (in, e.g., [Zlotnik and Li, 2014]). However, coupling of different subharmonically forced oscillators is not well understood.

Proposed Work

Questions:

- Can common driving selectively help or hinder mutual $N : M$ entrainment?
- Can coupling across timescales improve coherence within timescales?

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Pie-in-the-sky scientific question... Is the seven-day week adaptive?

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Start with a forced, nonlinear oscillator with stable limit cycle γ:

$$\dot{x} = f(x;u), \quad \dot{\gamma} = f(\gamma;0), \quad \gamma(t+T) = \gamma(t)$$

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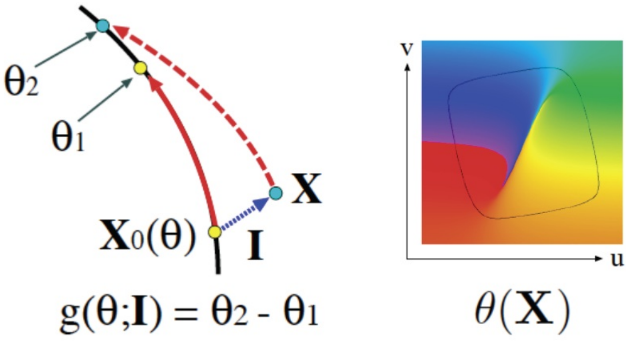

Nakao

Forcing and Coupling

Assuming that oscillators *respond* to forces according to $Z(\psi)$, *exert* forces according to $P(\psi)$, and are all subject to the same force $u = v(\Omega t)$,

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Changing coordinates, $\psi = \phi + \Omega t$, and averaging over $\Omega t \in [0, 2\pi)$:

$$\dot{\varphi}_i = \Delta\omega_i + \Lambda_v(\varphi_i) + \frac{K}{N}\sum_{j=1}^{N} G(\varphi_j - \varphi_i)$$

"Separate but Comparable"

Separation of timescales underlies much analysis:

- Center manifold reduction
- Model fast degrees of freedom as stationary noise: replace high-dimensional ODE by low-dimensional SDE